Buoyancy instabilities in galaxy clusters w. B
+
Solving MHD equations

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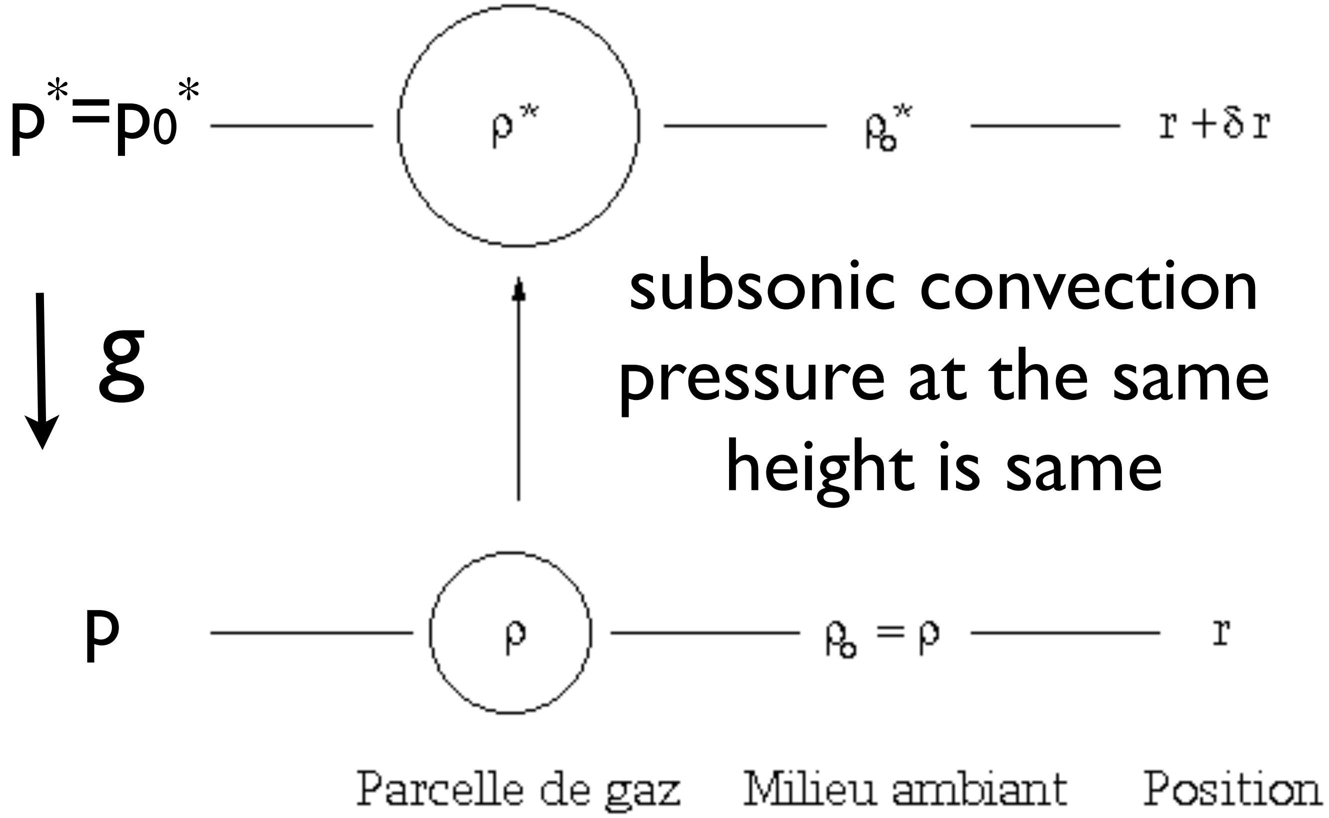
Outline

- instabilities due to anisotropic transport: MTI & HBI
- numerical solution of Euler eqs.: methods
- advection equation (model for hyperbolic eqs.): $\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0$
- extension to MHD: div(B)=0 constraint
- some publicly available codes

Schwarzschild convection



subsonic convection
pressure at the same
height is same

$$\rho_0^* = \rho_0 + \frac{d\rho}{dr}\delta r$$

$$\frac{p}{\rho^\gamma} = \frac{p^\star}{\rho^{\star\,\gamma}}$$ adiabatic blob

$$\rho^\star = \rho\left(\frac{p^\star}{p}\right)^{1/\gamma} \approx \rho\left(1 + \frac{1}{\gamma}\frac{d\ln p}{dr}\delta r\right)$$

$$\ddot{\delta r} + N^2\delta r = 0$$

buoyancy force: $(\rho^\star - \rho_0^\star)g = \frac{\rho g}{\gamma}\frac{d}{dr}\ln\left(\frac{p}{\rho^\gamma}\right)\delta r$

$$N^2 = \frac{g}{\gamma}\frac{d}{dr}\ln\left(\frac{p}{\rho^\gamma}\right)$$

stably stratified if entropy higher at top

Plasma Thermal Conductivity

diffusivity ($\mathrm{cm^2 s^{-1}}$) $\varkappa$: v_t x mfp; mfp~$1/(n\sigma)$; σ~$b^2 \ln\Lambda$; b~e^2/kT

diffusivity $\propto T^{5/2}/n$; e^-s conduct heat as they are 40 times faster than protons

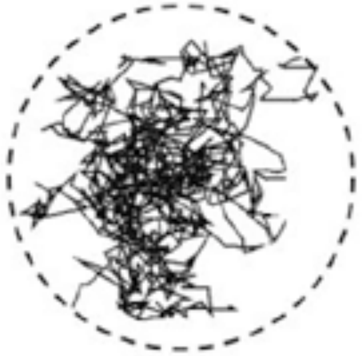
$$nT\frac{ds}{dt} = -\nabla \cdot \mathbf{Q} \qquad s = \frac{k_B}{\gamma - 1}\ln\left(\frac{p}{\rho^\gamma}\right)$$ entropy

the blob is not adiabatic for a conducting plasma!

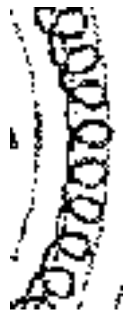
Plasma Thermal Conductivity

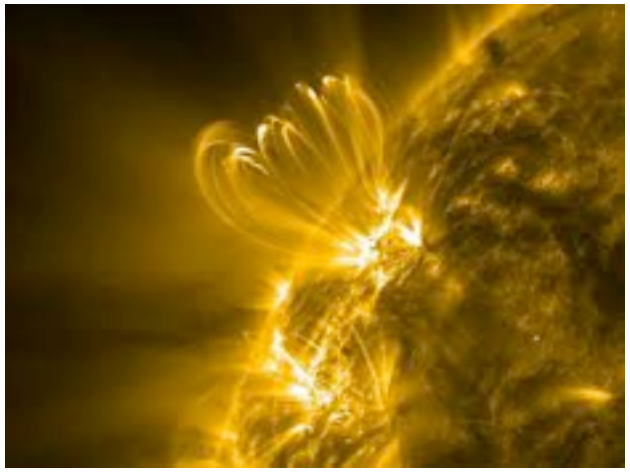
a tricky issue!

$$\mathbf{Q} = -\kappa \nabla T = -\chi n k_B \nabla T$$ for unmagnetized plasma



$$\mathbf{Q} = -\kappa \hat{b} \nabla_{\|} T = -\kappa \hat{b}(\hat{b} \cdot \nabla) T$$ for magnetized plasma



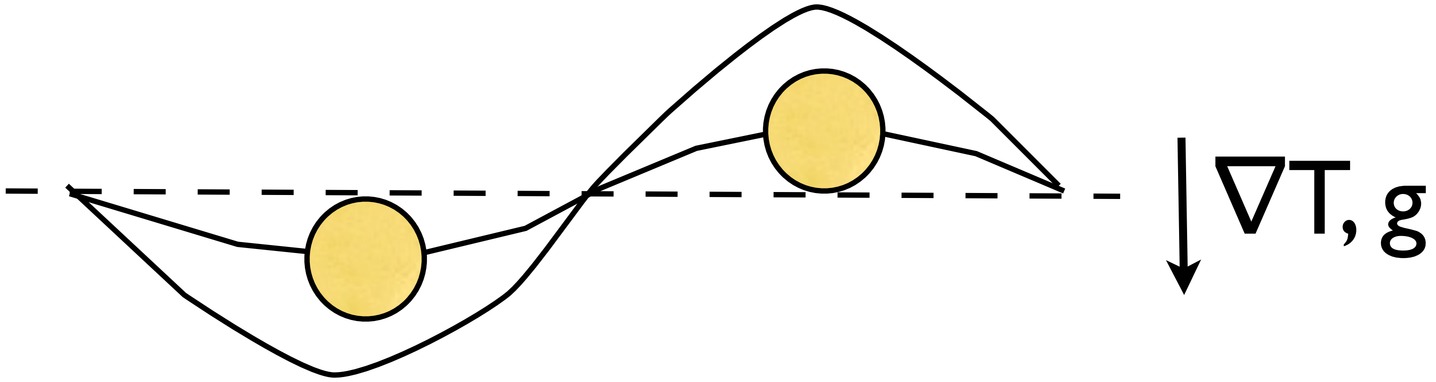


particles move along B w. small Larmor radii but diffuse along B with a path length of mfp; mfp>>ρ_L

$$D_{\|} \sim v_t^2/\nu \gg D_{\perp} \sim \rho_L^2 \nu$$ true for all transport coeffts.

All this is fine for a given B, but B changes because of plasma currents, small scale instabilities! Observed perp. transport is enhanced. This is the key problem of tokamaks.

Modification of Schwarzschild criterion



intracluster medium:
plasma that fills the space between galaxies in
Braginskii transport limit:
$\rho_L \ll$ mfp $< L$

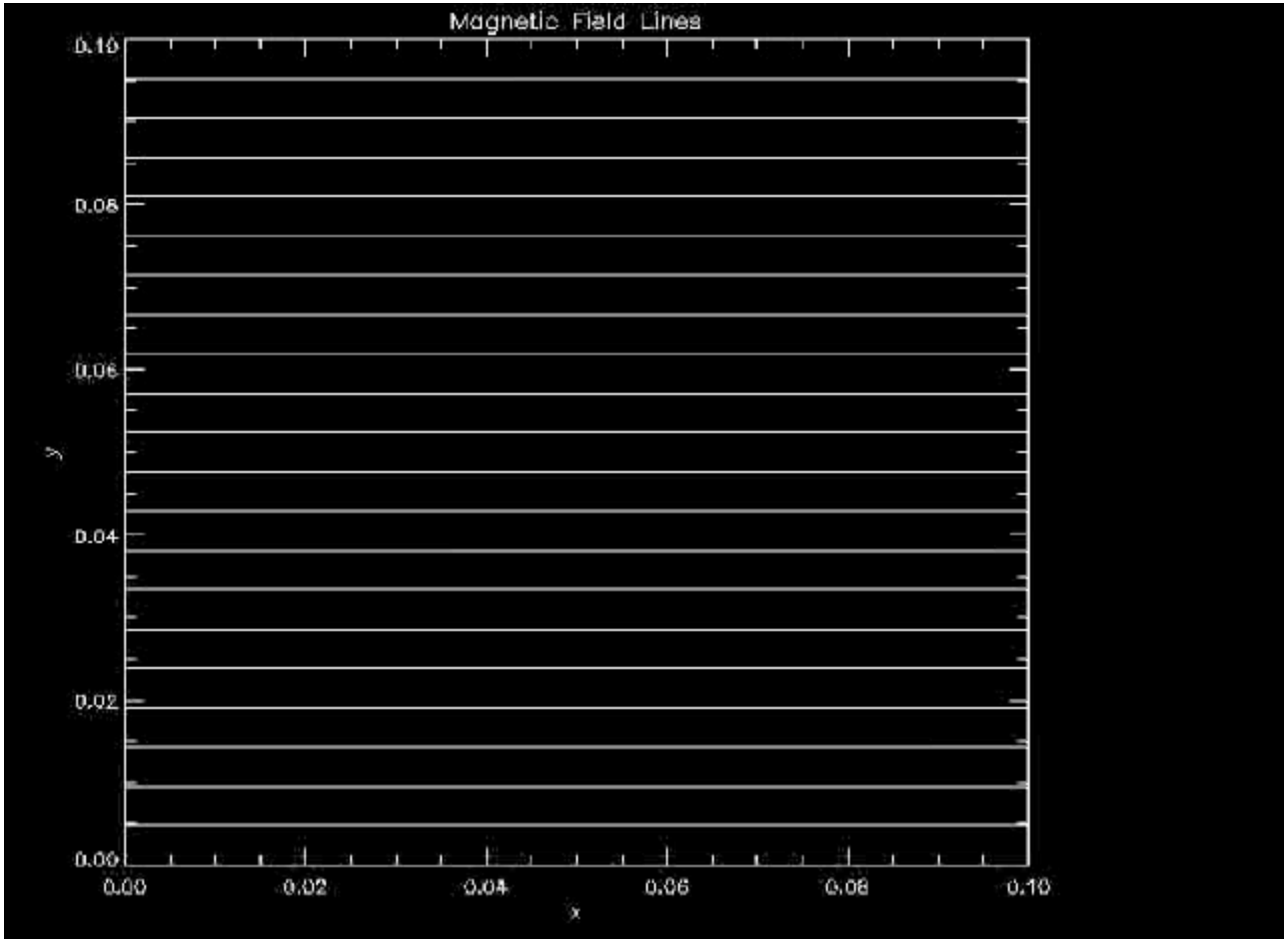
weak B: only role is aniso. cond.
$t_{cond} \ll t_{buoy}$

=> flow is unstable
if $dT/dz < 0$ even if $ds/dz > 0$!
a similar instability for $dT/dz > 0$!

[Balbus 2000, Quataert 2008]

Magnetothermal Instability (MTI)

[from Ian Parrish's website]



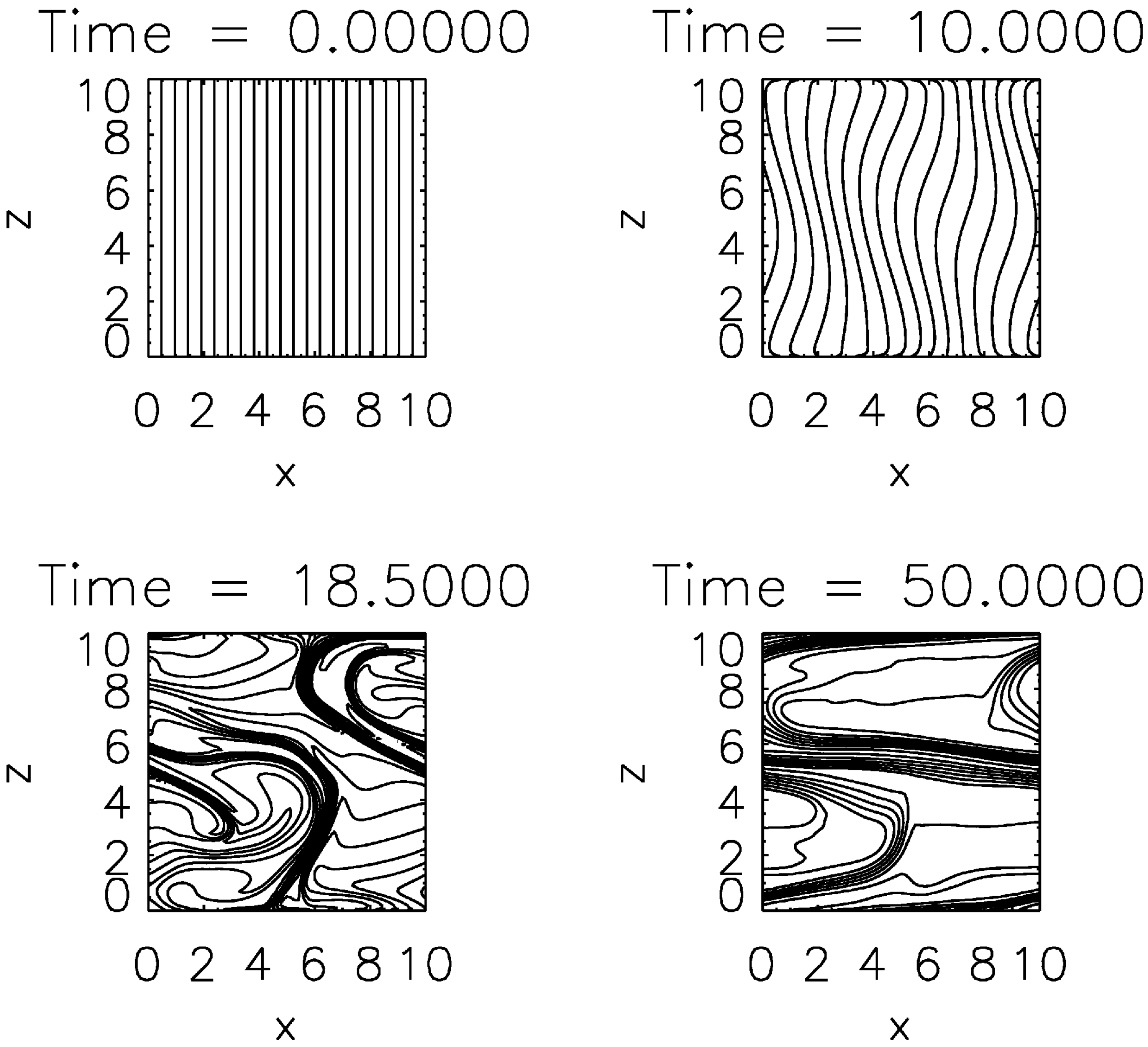
temperature maximum at bottom

reflective BCs at top and bottom

outflow more realistic

Heat-flux Buoyancy Instability (HBI)

[Parrish & Quataert 2008]

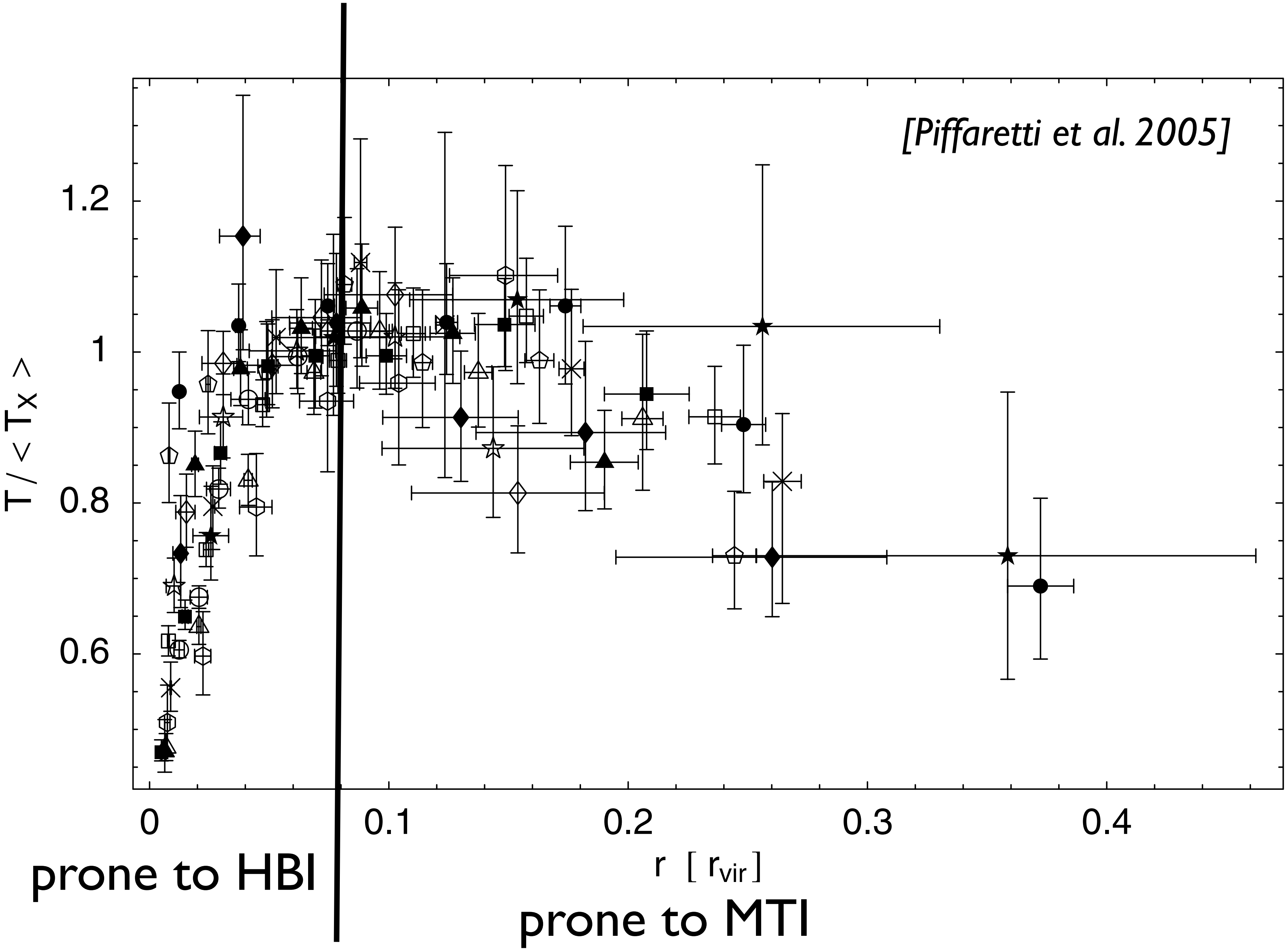


temperature
minimum at bottom

HBI reorients field lines
perpendicular to ∇T

cuts off thermal conduction

Temperature profiles in galaxy clusters

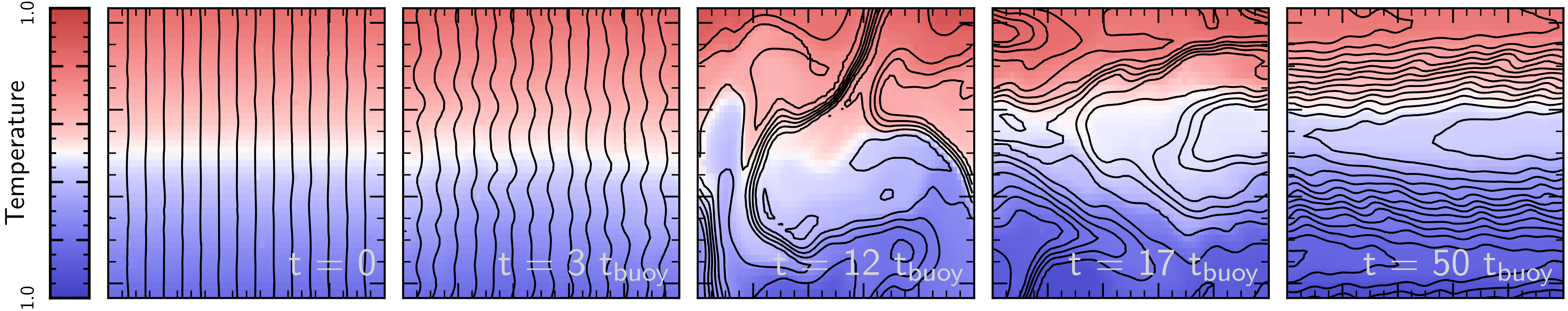


prone to HBI

prone to MTI

Implications for galaxy clusters?

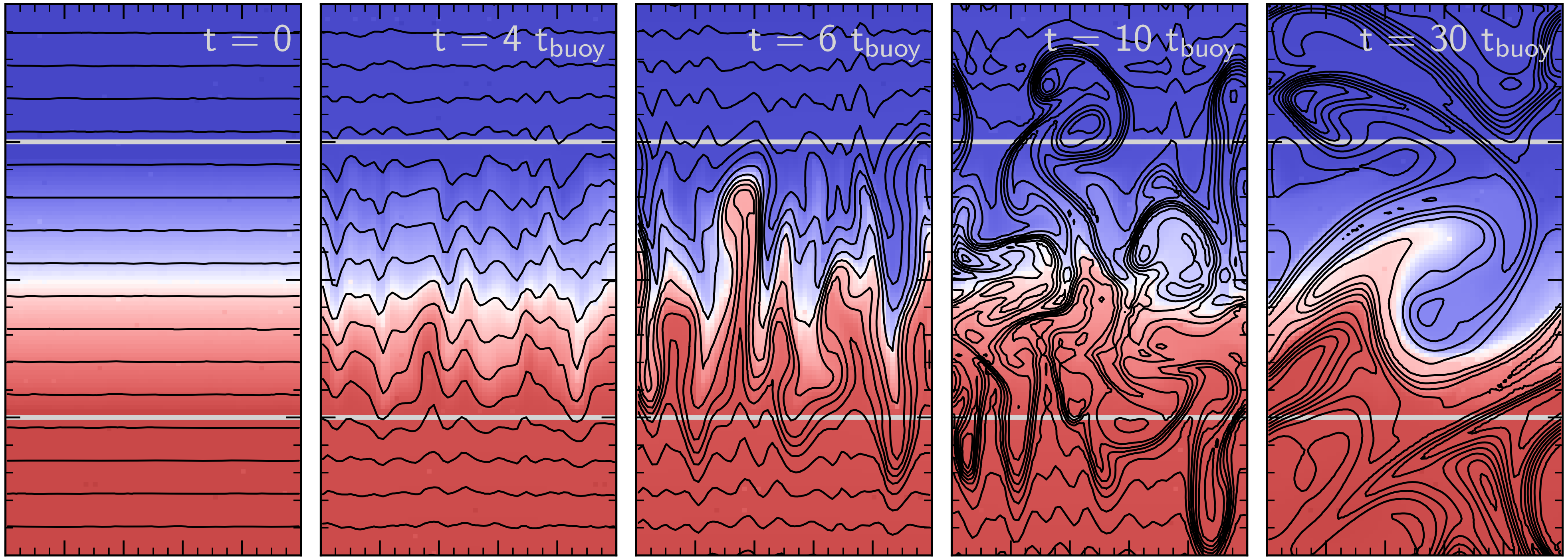
thermal conduction can no longer bring heat out from large radii => cooling flow



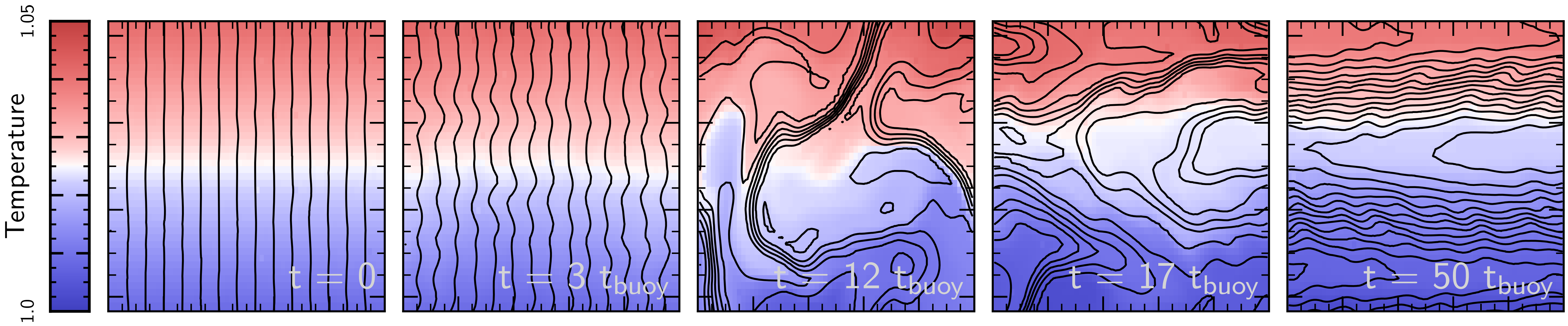
HBI saturates by reorienting B but MTI is a robust instability

[McCourt et al. 2010]

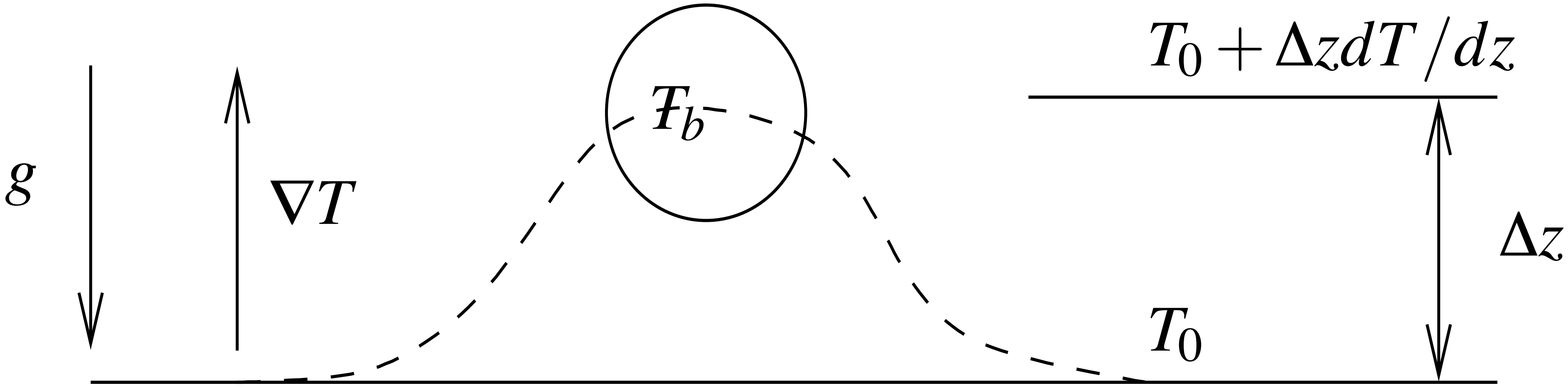
can stir clusters
at large radii
bias mass
measurements
via SZ that
assume HSE



Is HBI saturated state stable?



HBI saturates by reorienting B but MTI does not. MTI sustained as horizontal motions don't require energy & create horizontal B.



blob experiences a restoring force once B is reoriented; this stabilization can be overcome by a turbulent velocity of ~ 100 km/s

Gist of MTI/HBI

- magnetic tension & Braginskii viscosity further suppress these
- turbulence from AGN/mergers can overwhelm them
- interesting physics w. implications on mixing, etc.
- fundamental Qs about heat/momentum transport in collisionless plasmas: kinetic instabilities

Solving MHD equations on massive clusters

There are several approaches for solving MHD equations:
finite difference, SPH, finite element, spectral, etc.

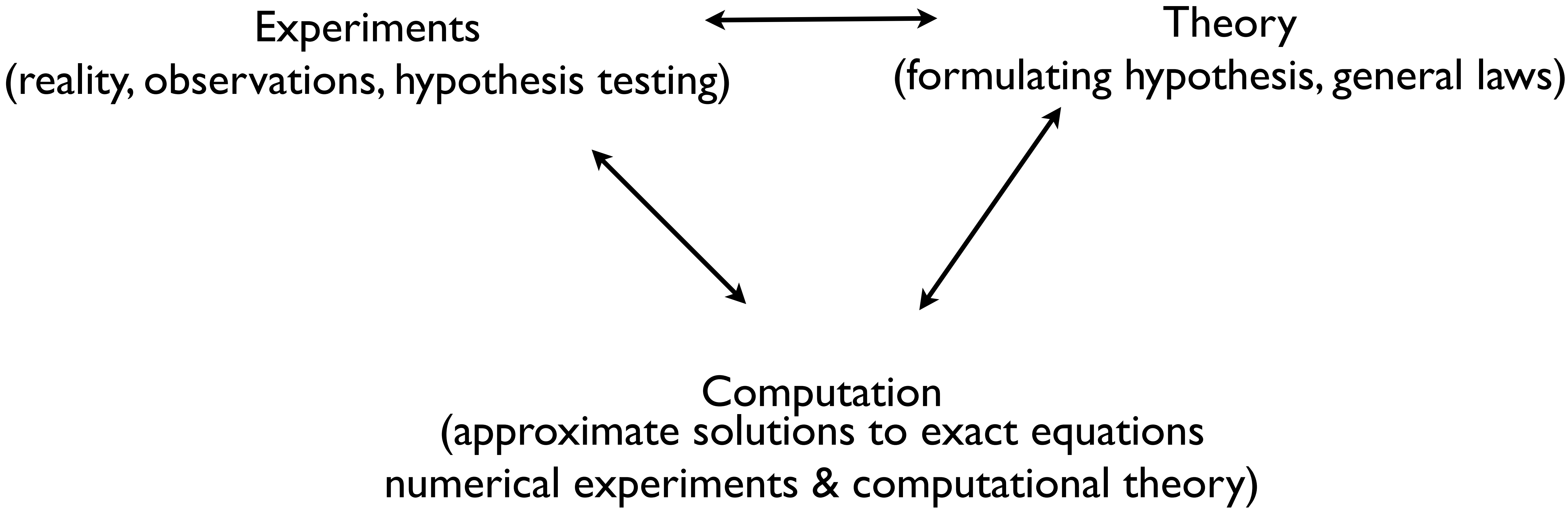
Here I will talk about the most mature and widely
applicable finite volume schemes

borrowed material from Greg Hammett's talk

References:
Finite Volume Methods for Hyperbolic Problems, R J Leveque, Cambridge
Stone & Norman ZEUS papers 1992
all popular code (ATHENA, PLUTO, RAMSES) websites/user guides

Pillars of modern science

[adapted from Greg Hammett's slides]



Experiments
(reality, observations, hypothesis testing)

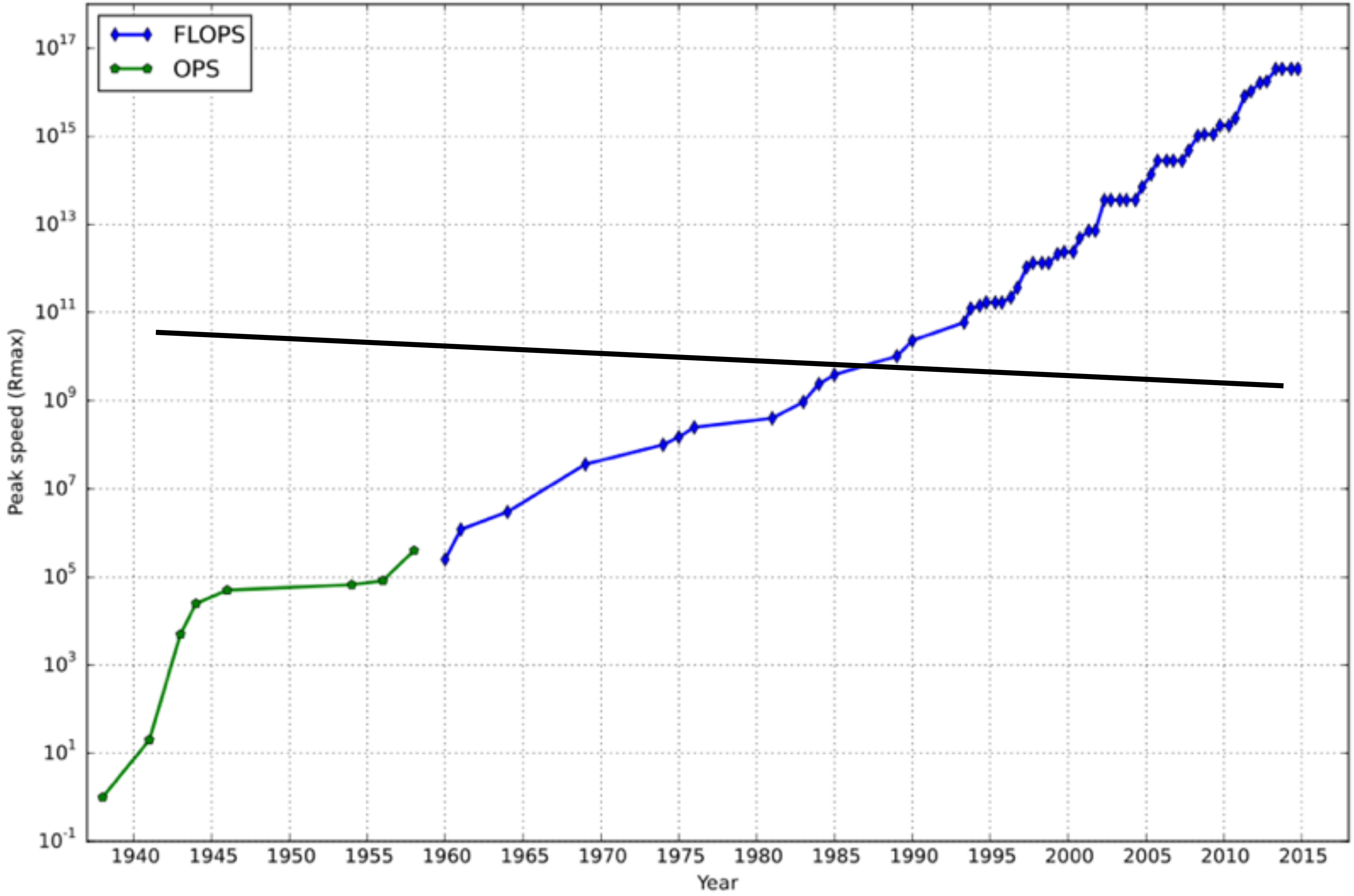
Theory
(formulating hypothesis, general laws)

Computation
(approximate solutions to exact equations
numerical experiments & computational theory)

Especially useful in astrophysics: highly nonlinear systems; ideas must be tested numerically

Why care about HPC?

top supercomputer speeds as a function of time from Wikipedia



MHD equations

conservative form: modern finite volume approach

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0,$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}\boldsymbol{v} - \boldsymbol{B}\boldsymbol{B} + \mathbf{P}^*) = 0,$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + P^*)\boldsymbol{v} - \boldsymbol{B}(\boldsymbol{B} \cdot \boldsymbol{v})] = 0,$$

$$\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = 0,$$

$$P^* = P + B^2/2$$

$$E = \frac{P}{\gamma - 1} + \frac{1}{2}\rho v^2 + \frac{B^2}{2},$$

operator splitting: ZEUS approach

source terms

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{\nabla} p - \rho \boldsymbol{\nabla} \Phi - \boldsymbol{\nabla} \cdot \mathbf{Q},$$

$$\frac{\partial e}{\partial t} = -p \boldsymbol{\nabla} \cdot \boldsymbol{v} - \mathbf{Q} : \boldsymbol{\nabla} \boldsymbol{v},$$

transport terms

$$\frac{d}{dt} \int_V \rho \, dV = -\int_{dV} \rho (\boldsymbol{v} - \boldsymbol{v}_g) \cdot d\boldsymbol{S},$$

$$\frac{d}{dt} \int_V \rho \boldsymbol{v} \, dV = -\int_{dV} \rho \boldsymbol{v} (\boldsymbol{v} - \boldsymbol{v}_g) \cdot d\boldsymbol{S},$$

$$\frac{d}{dt} \int_V e \, dV = -\int_{dV} e (\boldsymbol{v} - \boldsymbol{v}_g) \cdot d\boldsymbol{S},$$

Operator splitting

$$\frac{\partial y}{\partial t} = \mathscr{L}(y)$$

say you want to apply cooling
losses to the energy equation,
energy source term due to supernovae, etc.

$$\mathscr{L}(y) = \mathscr{L}_1(y) + \mathscr{L}_2(y) + \cdots$$

$$(y^1 - y^0)/\Delta t = L_1(y^0) ,$$

simple way of adding additional physics
to available code

$$(y^2 - y^1)/\Delta t = L_2(y^1) ,$$

ZEUS use operator splitting. Source terms use FTCS
but transport terms are solved in a conservative upwind form

$$(y^3 - y^2)/\Delta t = \cdots ,$$

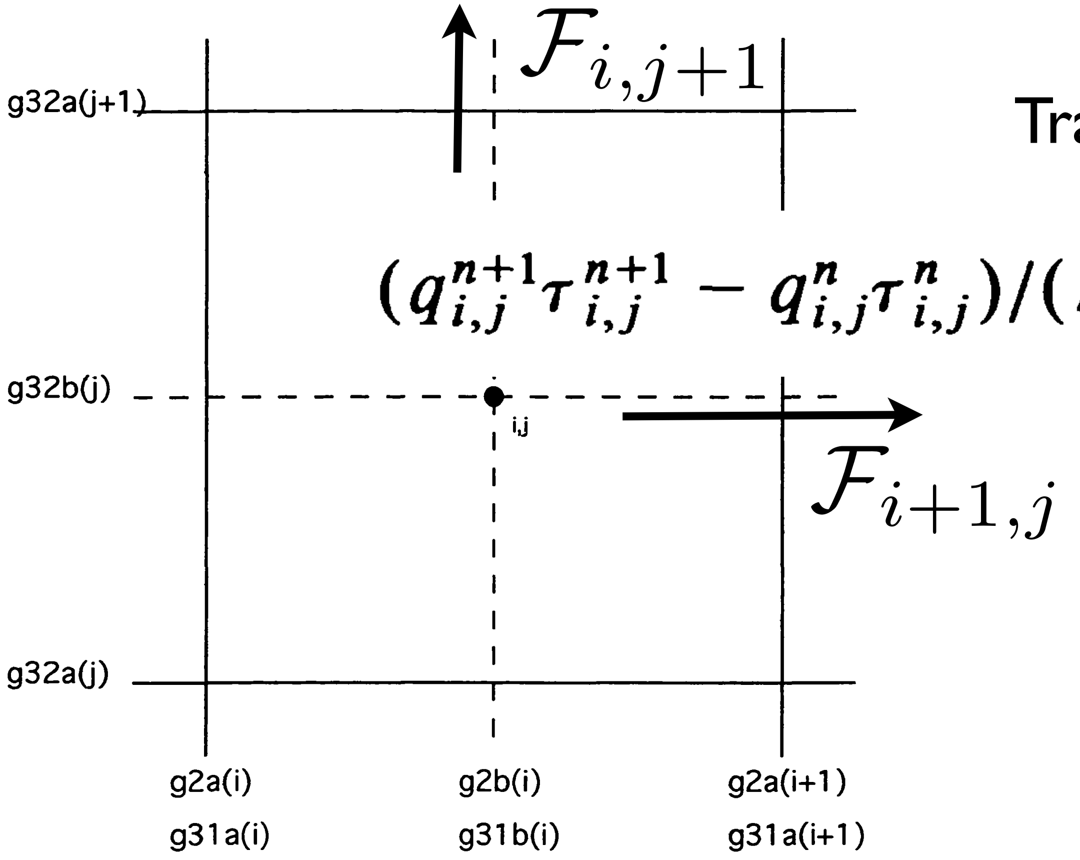
doesn’t conserve total momentum and total energy to machine precision

ZEUS example

Source step:

$$\frac{v1^{n+a}_{i,j} - v1^{n}_{i,j}}{\Delta t} = -\frac{p^{n}_{i,j} - p^{n}_{i-1,j}}{dx1b_i(d^{n}_{i,j} + d^{n}_{i-1,j})/2} - \frac{\Phi^{n}_{i,j} - \Phi^{n}_{i-1,j}}{dx1b_i}$$

$$\frac{v2^{n+a}_{i,j} - v2^{n}_{i,j}}{\Delta t} = -\frac{p^{n}_{i,j} - p^{n}_{i,j-1}}{g2b_i\, dx2b_j(d^{n}_{i,j} + d^{n}_{i,j-1})/2} - \frac{\Phi^{n}_{i,j} - \Phi^{n}_{i,j-1}}{g2b_i\, dx2b_j}$$



Transport step: this step is conservative

$$(q^{n+1}_{i,j}\tau^{n+1}_{i,j} - q^{n}_{i,j}\tau^{n}_{i,j})/(\Delta t) = -(\mathscr{F}^{1}_{i+1,j} - \mathscr{F}^{1}_{i,j} + \mathscr{F}^{2}_{i,j+1} - \mathscr{F}^{2}_{i,j})^{n+1/2},$$

time averaged interpolated value

$$(\mathscr{F}^{k}_{i,j})^{n+1/2} = \langle\ vk_{i,j}\, \underset{\downarrow}{q^{*}_{k,i,j}}\tilde{A}_{k,i,j}\rangle$$

can also do directional splitting

Advection equation

how to get the face-centered, time centered interpolated quantity? Key difficulty in CFD.

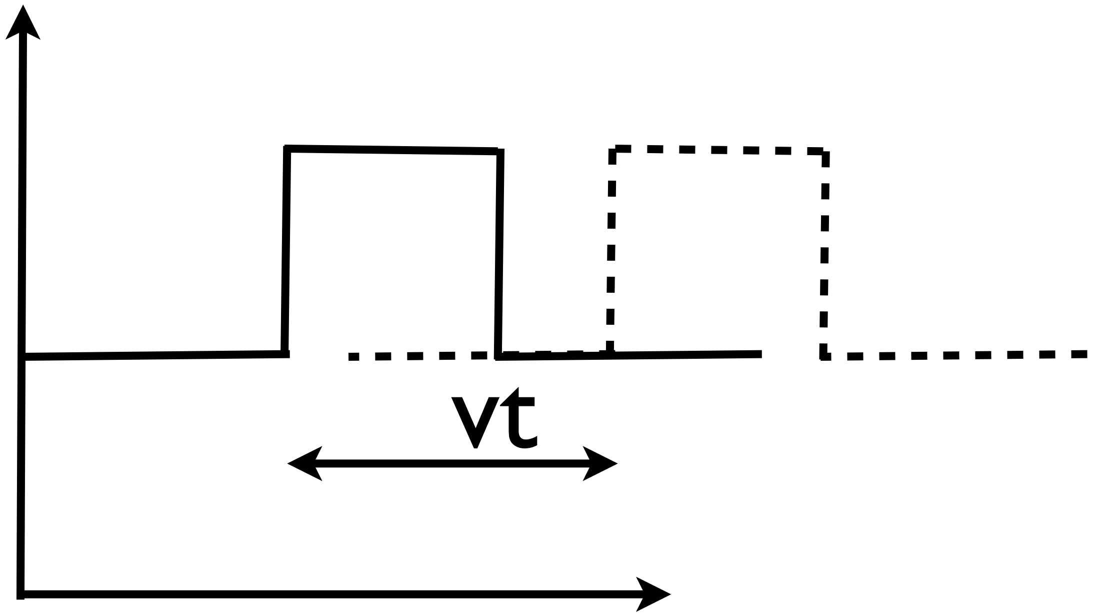
A model equation for hyperbolic equations, the advection equation:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0$$

v the constant advection velocity

$\mathcal{F} = vf$ flux

$f(x,t) = f(x - vt, t = 0)$ just the initial condition shifted in space by vt



Upwinding

Lax Wendroff scheme: (second order in x&t)

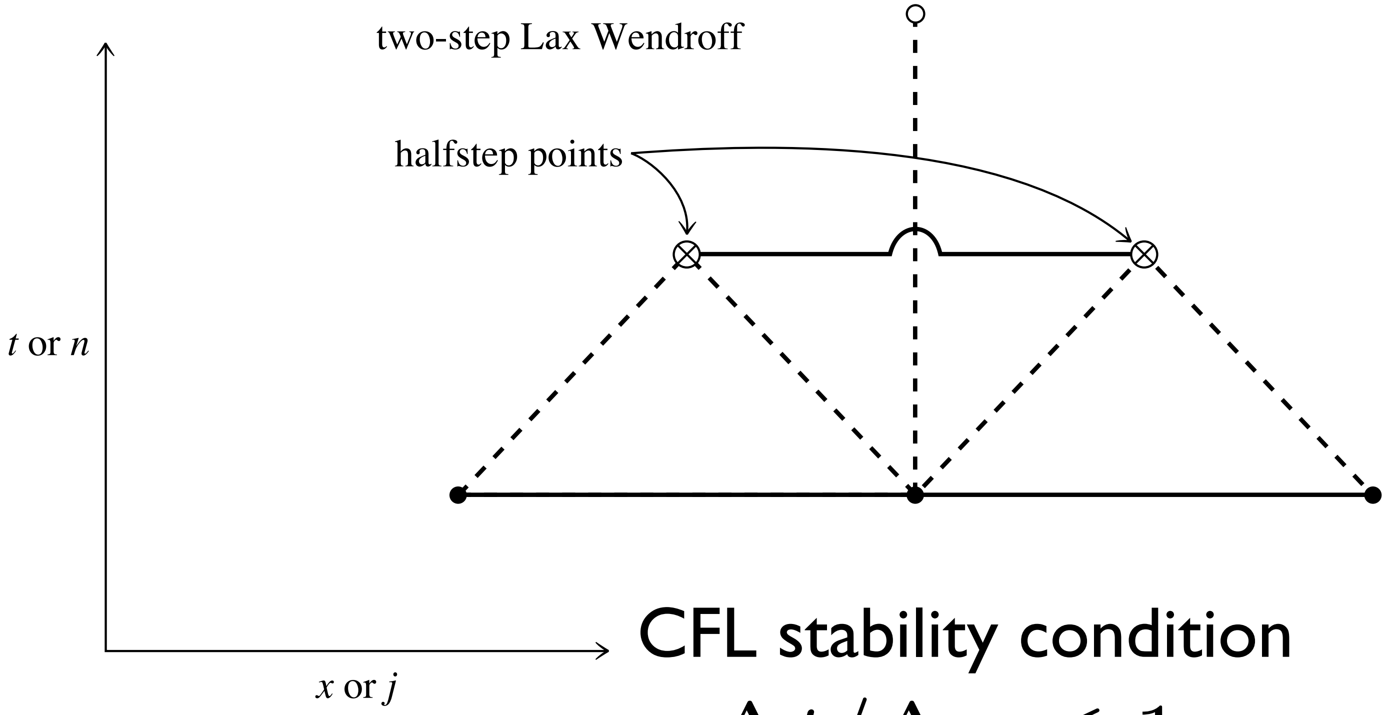
$$f_{i+1/2}^{n+1/2} = \frac{1}{2}(f_{i+1}^n + f_i^n) - \frac{\Delta t}{2\Delta x}(\mathcal{F}_{i+1}^n - \mathcal{F}_i^n)$$

$$f_i^{n+1} = f_i^n - \frac{\Delta t}{\Delta x}(\mathcal{F}_{i+1/2}^{n+1/2} - \mathcal{F}_{i-1/2}^{n+1/2})$$

information propagates in one direction; scheme shouldn't be symmetric in x

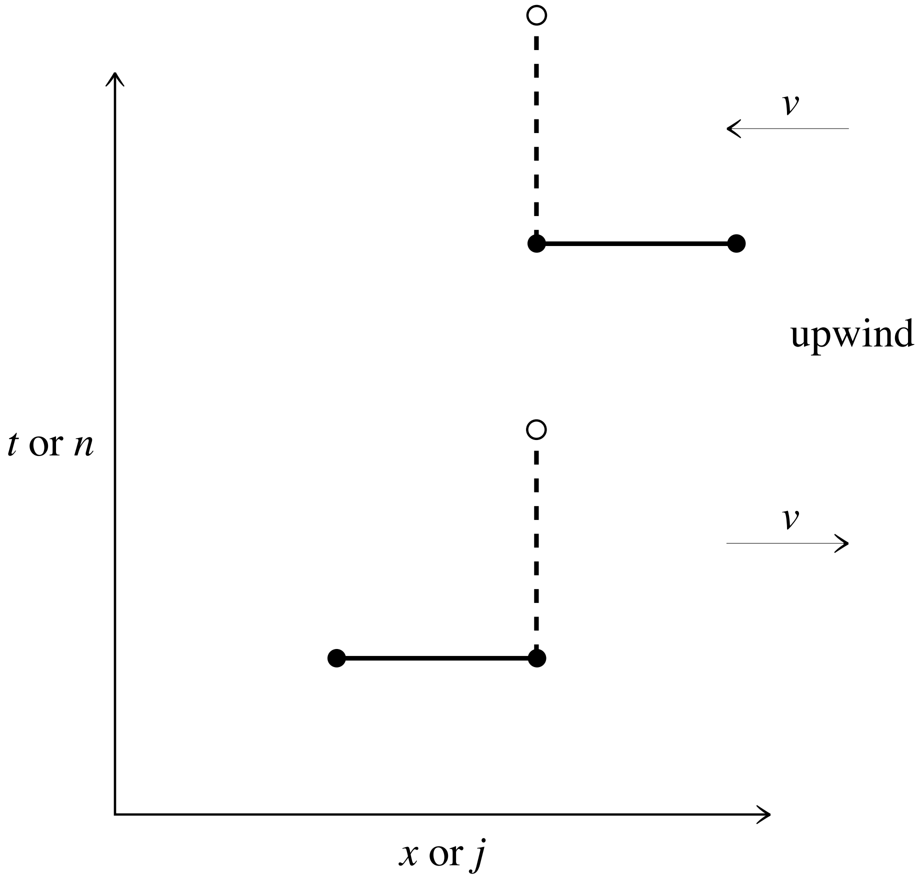
$$\frac{f_i^{n+1} - f_i^n}{\Delta t} = -v\frac{f_i^n - f_{i-1}^n}{\Delta x}, \text{ if } v \geq 0$$

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} = -v\frac{f_{i+1}^n - f_i^n}{\Delta x}, \text{ if } v < 0$$

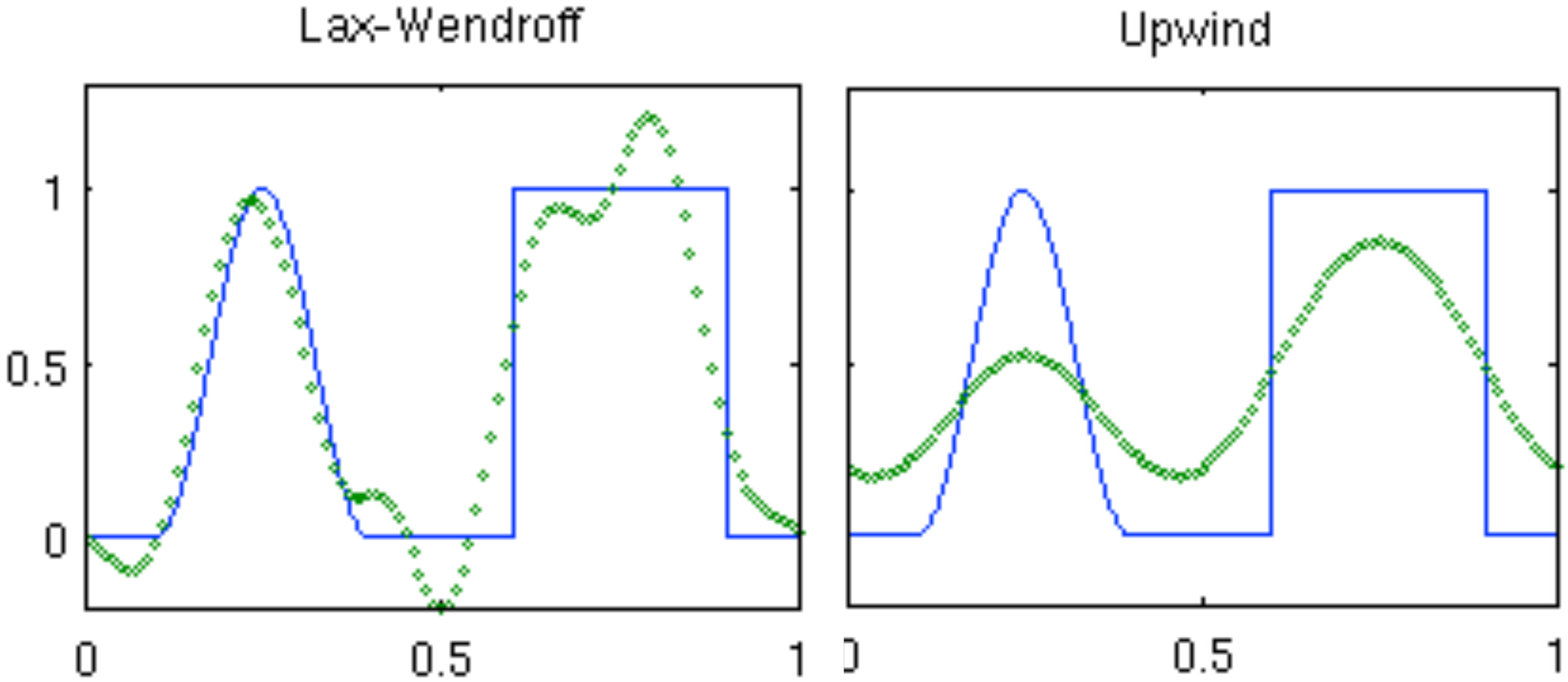


CFL stability condition

$$v\Delta t/\Delta x \leq 1$$



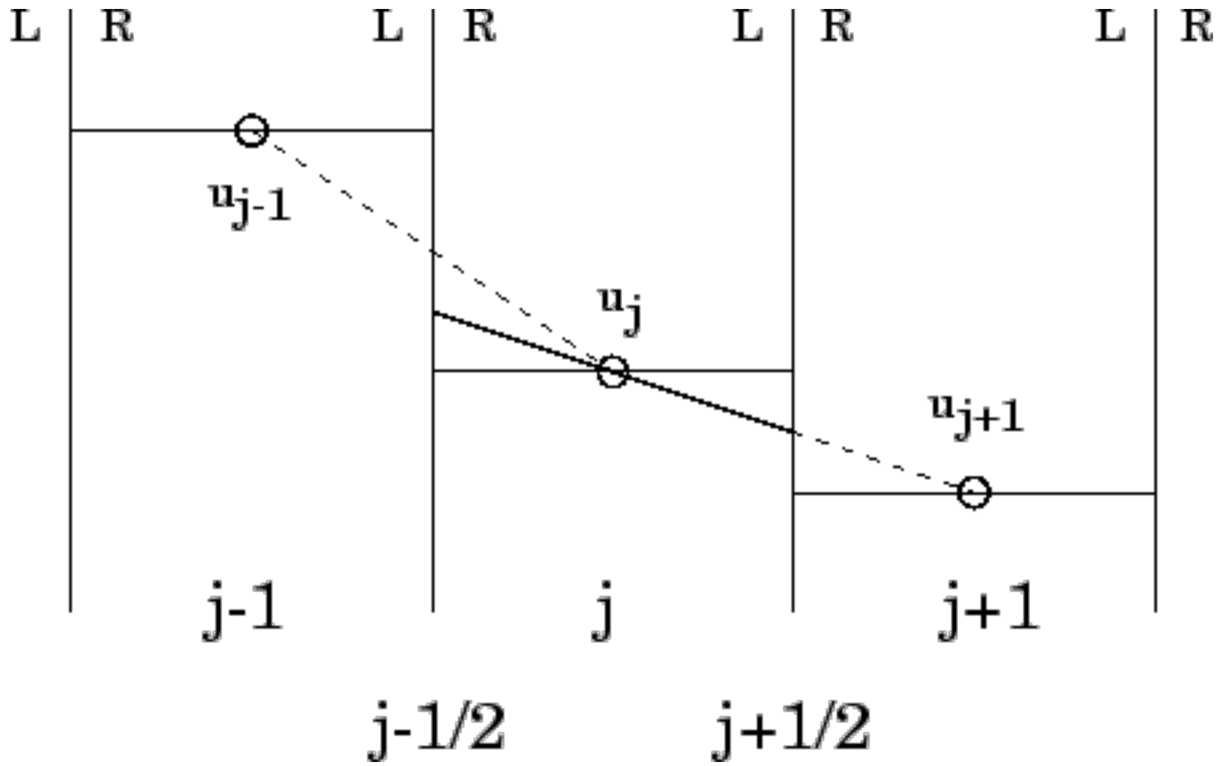
Comparison



L-W doesn't preserve monotonicity property (has oscillations/phase errors)!
Simple upwind is too diffusive

Physical solutions of Euler equations allow for shocks so this is not an academic exercise

Improving upwind method



use a higher order reconstruction, like linear or parabolic, instead of piecewise constant

Reconstruct f(x) in each cell, extrapolate to right boundary (for upwind flux if *v>0*):

$$f_{i+1/2} = f_i + s_i \Delta x/2$$

Piecewise constant = 1st order upwind :

$$s_i = 0$$

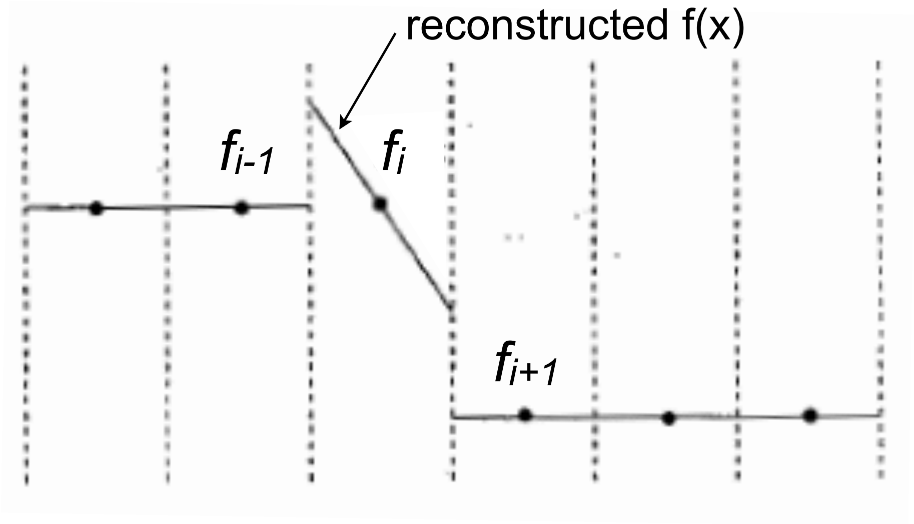
Downwind slope (centered 2cd order flux):

$$s_i = s_{i+1/2} = \frac{f_{i+1} - f_i}{\Delta x}$$

Upwind slope (upwind-biased 2cd order flux):

$$s_i = s_{i-1/2} = \frac{f_i - f_{i-1}}{\Delta x}$$

REA approach

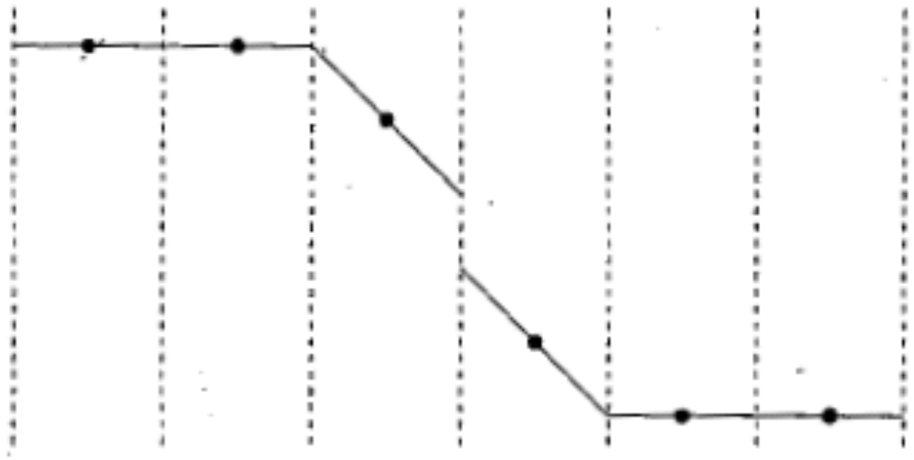


Reconstruction Evolution Average (REA)

i) Reconstruct a piecewise polynomial defined for all x (should have the same cell average)

ii) Evolve the hyperbolic equation exactly/approximately to get updated solution at t^{n+1}(Riemann problem)

iii) Average solution at t^{n+1} to get new cell-averaged values

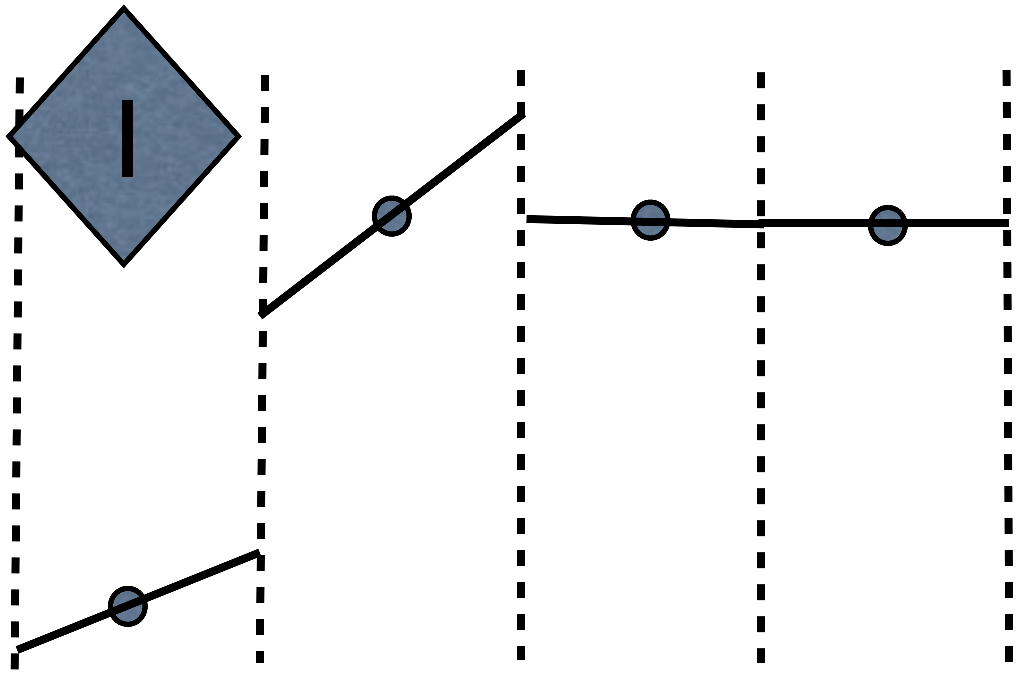


Van Leer’s (MC) limiter:
“Monotonized Central”

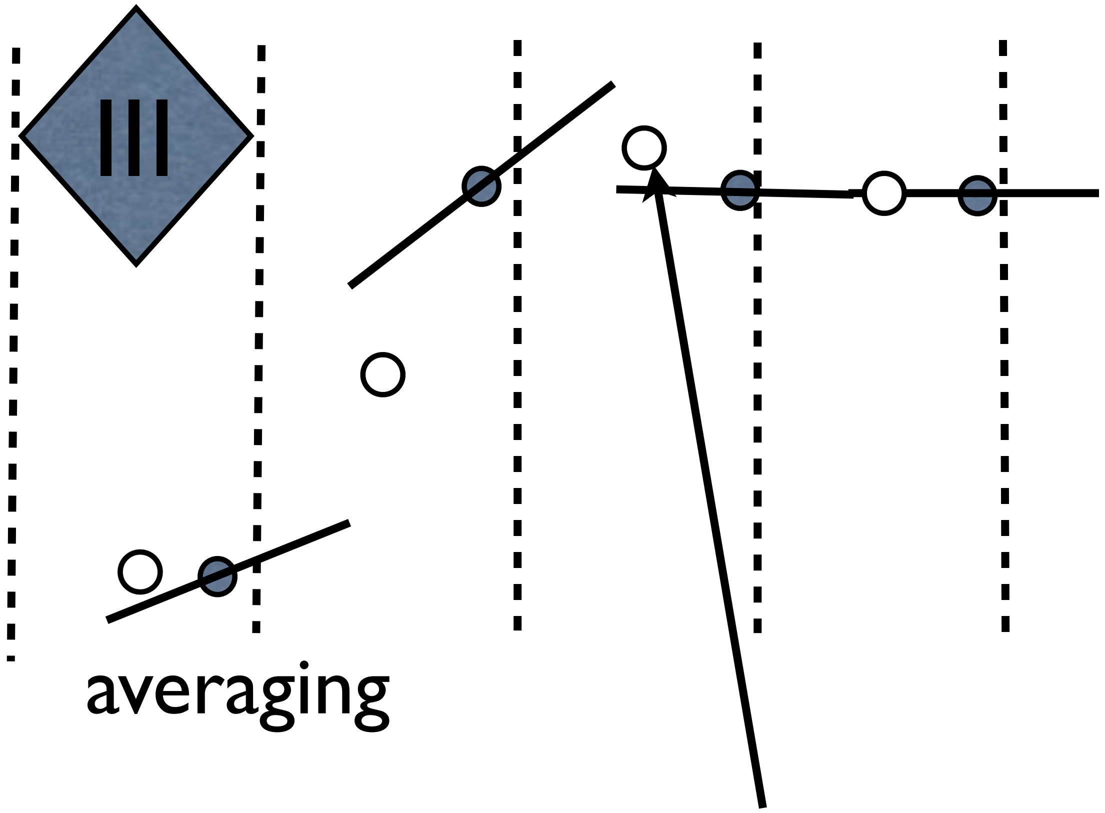
$$s_i = \mathrm{minmod}\left(\frac{s_{i-1/2}+s_{i+1/2}}{2}, 2s_{i-1/2}, 2s_{i+1/2}\right)$$

(minmod(a,b,c) = sign(a) * min(|a|,|b|,|c|) if a, b, and c all have same sign, otherwise minmod(a,b,c)=0.)

Limiters in advection

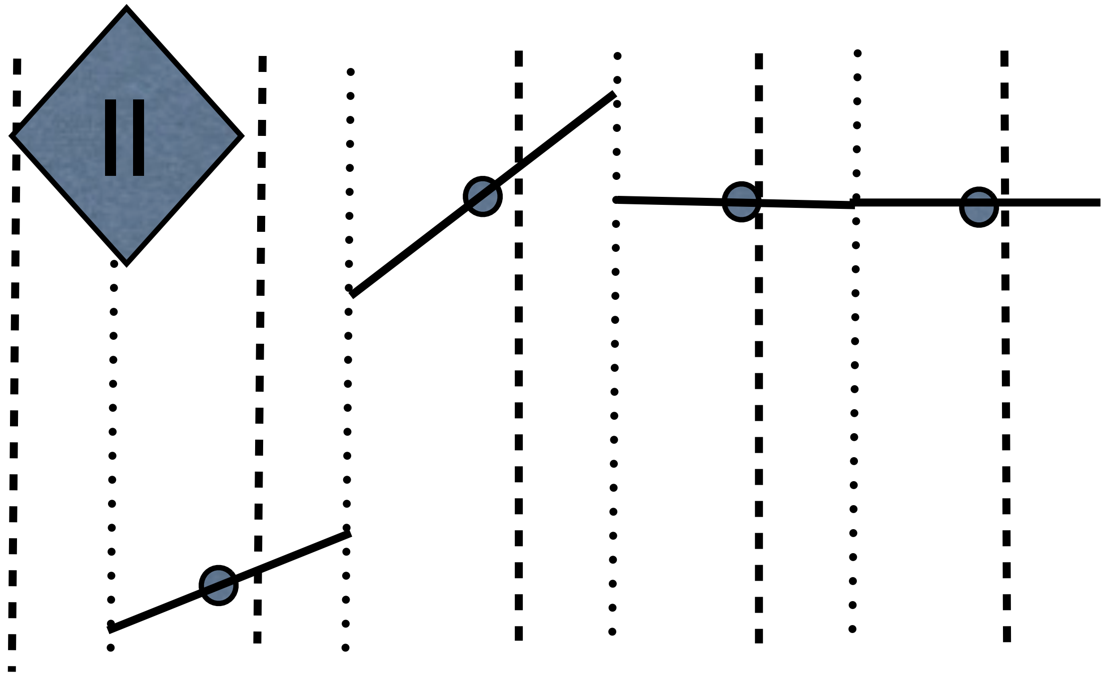


linear reconstruction: AM of slopes



averaging

non-monotonic
causes oscillations
at discontinuities

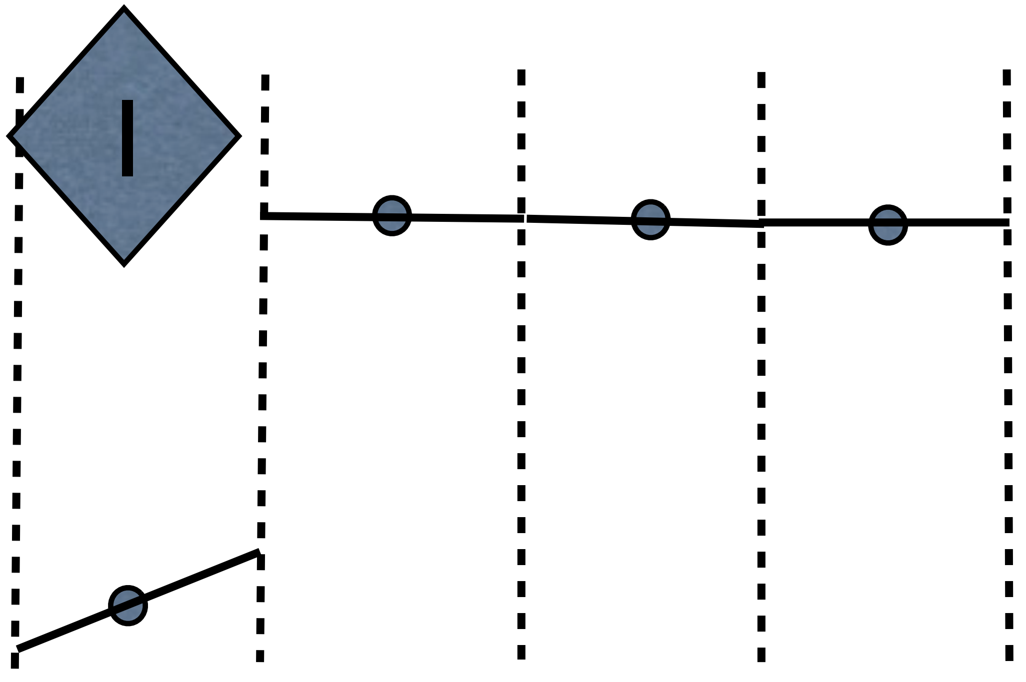


advection with v>0

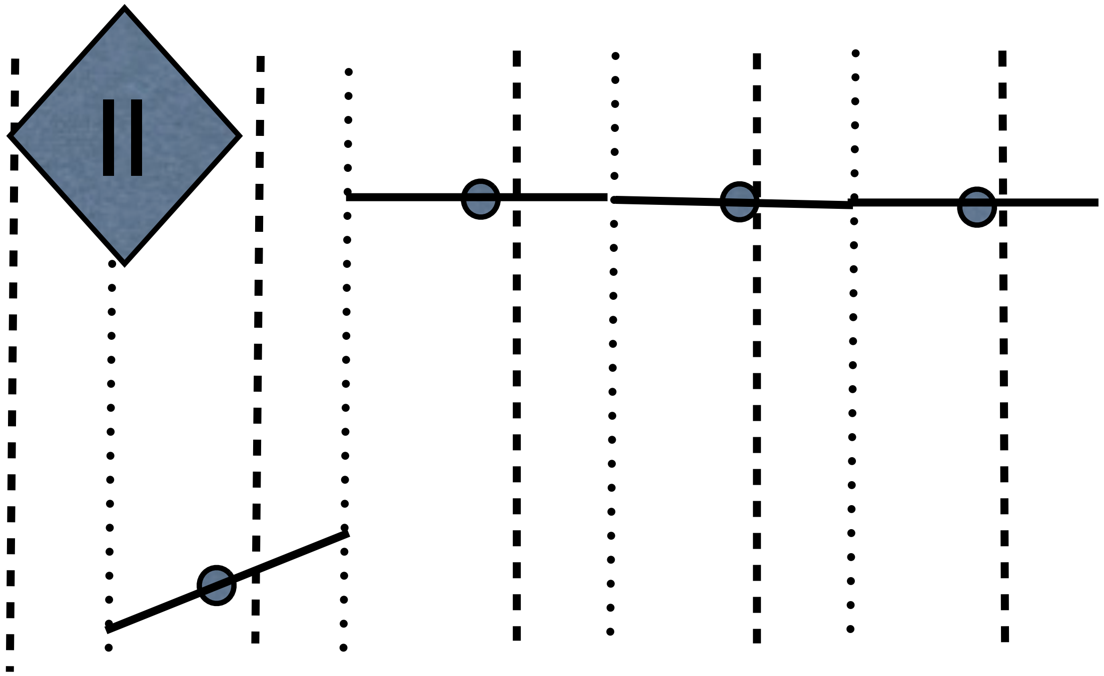
$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0$$

advection equation: REA approach

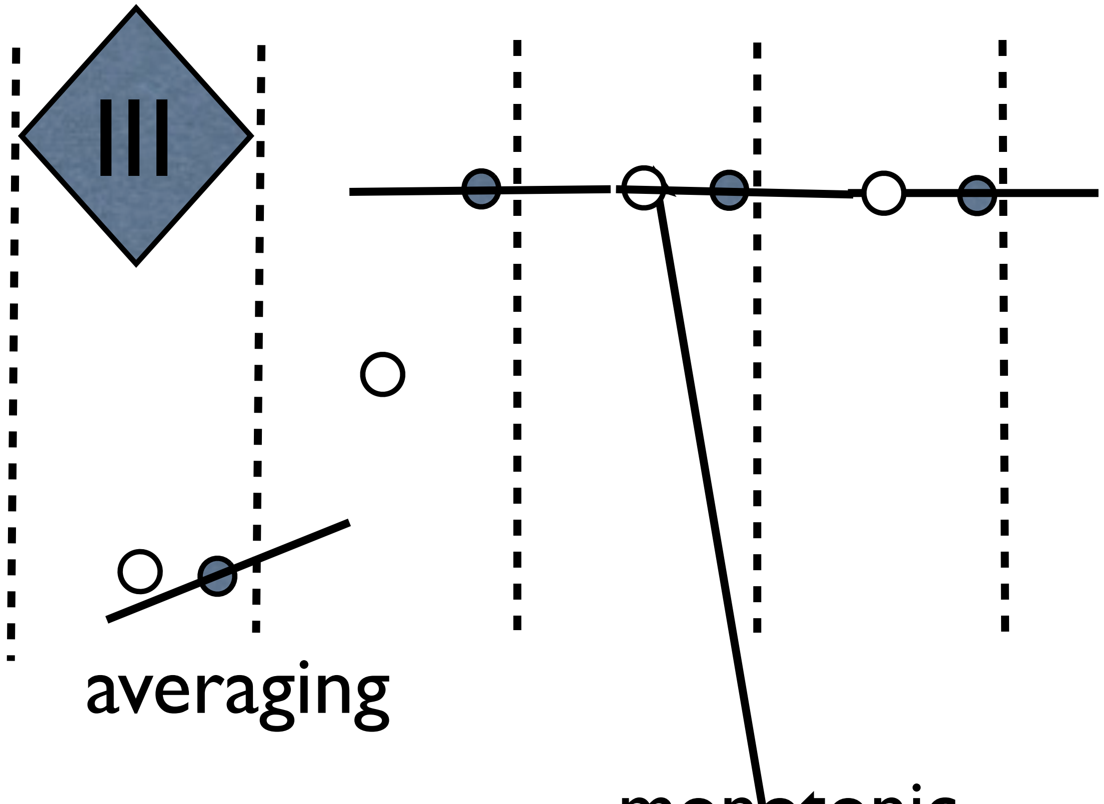
Limiters in advection



linear reconstruction: $\mathcal{L}$ of slopes



advection with v>0



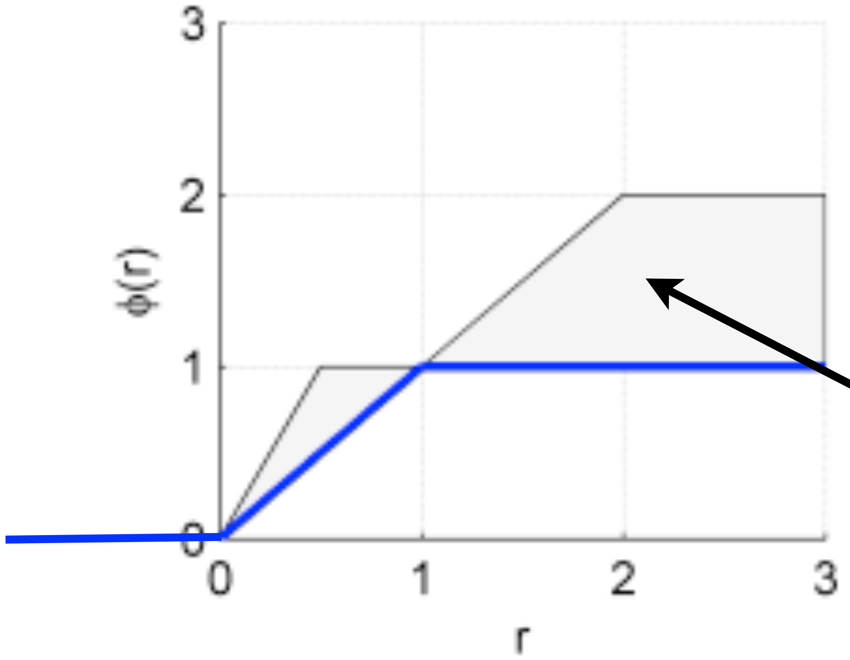
averaging

monotonic
prevents oscillations
at discontinuities if
limited reconstruction

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0$$

advection equation: REA approach

Minmod limiter

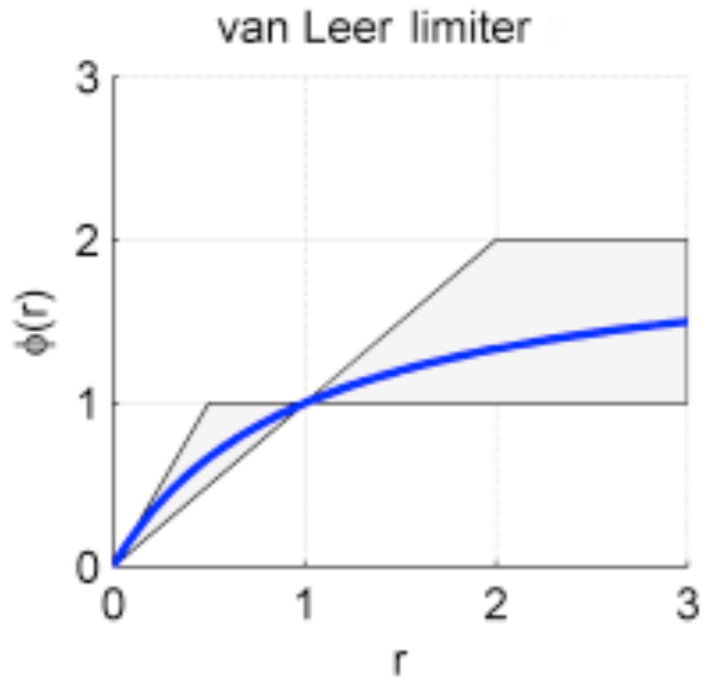


$$\mathcal{L}(a,b) = a\phi(1, r = b/a) = b\phi(1, r = a/b)$$

$\mathcal{L}$=0 if arguments have opposite sign

grey: monotonicity zone

minmod(a,b)=0 if ab<=0, min(a,b) if ab>0

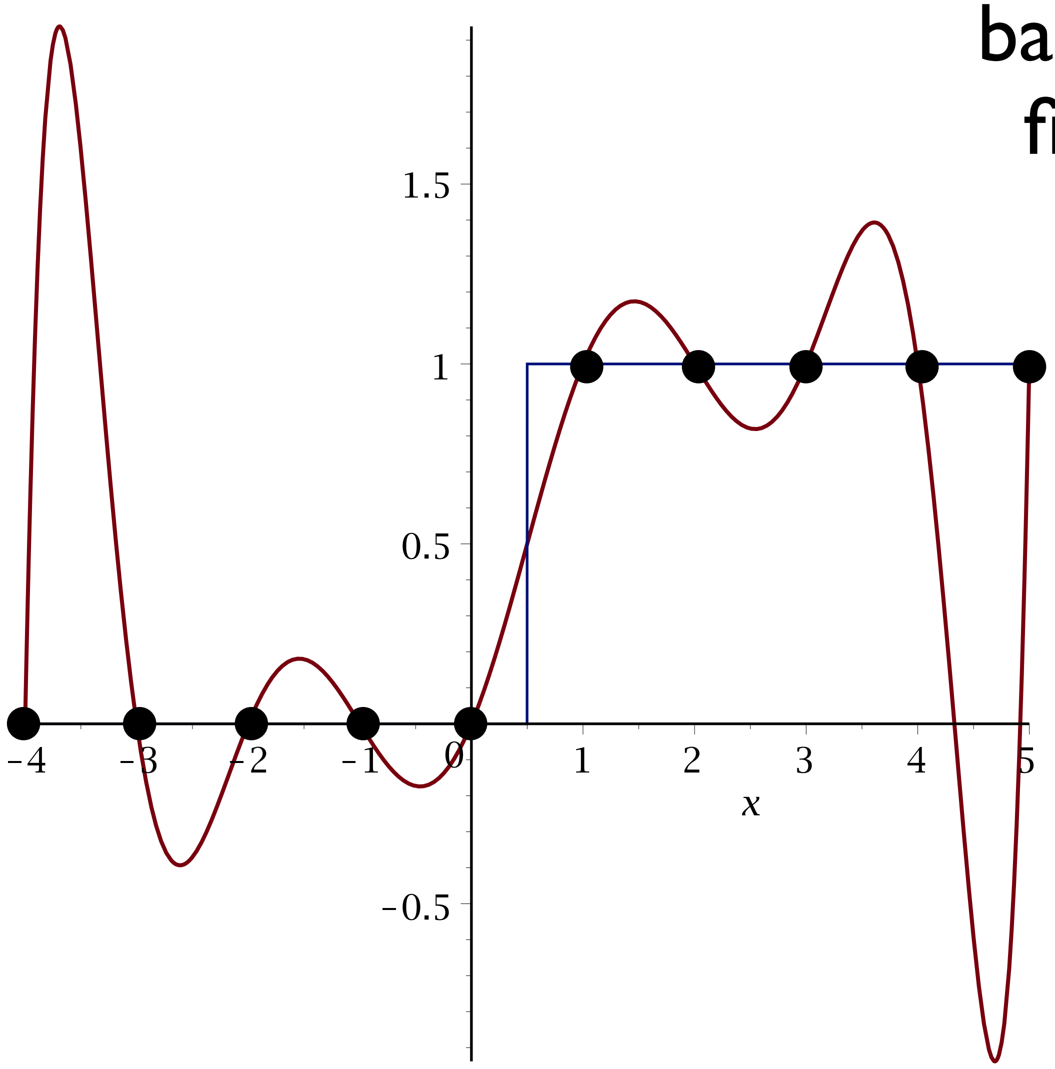


smoother, higher order accuracy, still nonlinear

vanleer(a,b)=0 if ab<=0, 2ab/(a+b) if ab>0

Very high order reconstruction

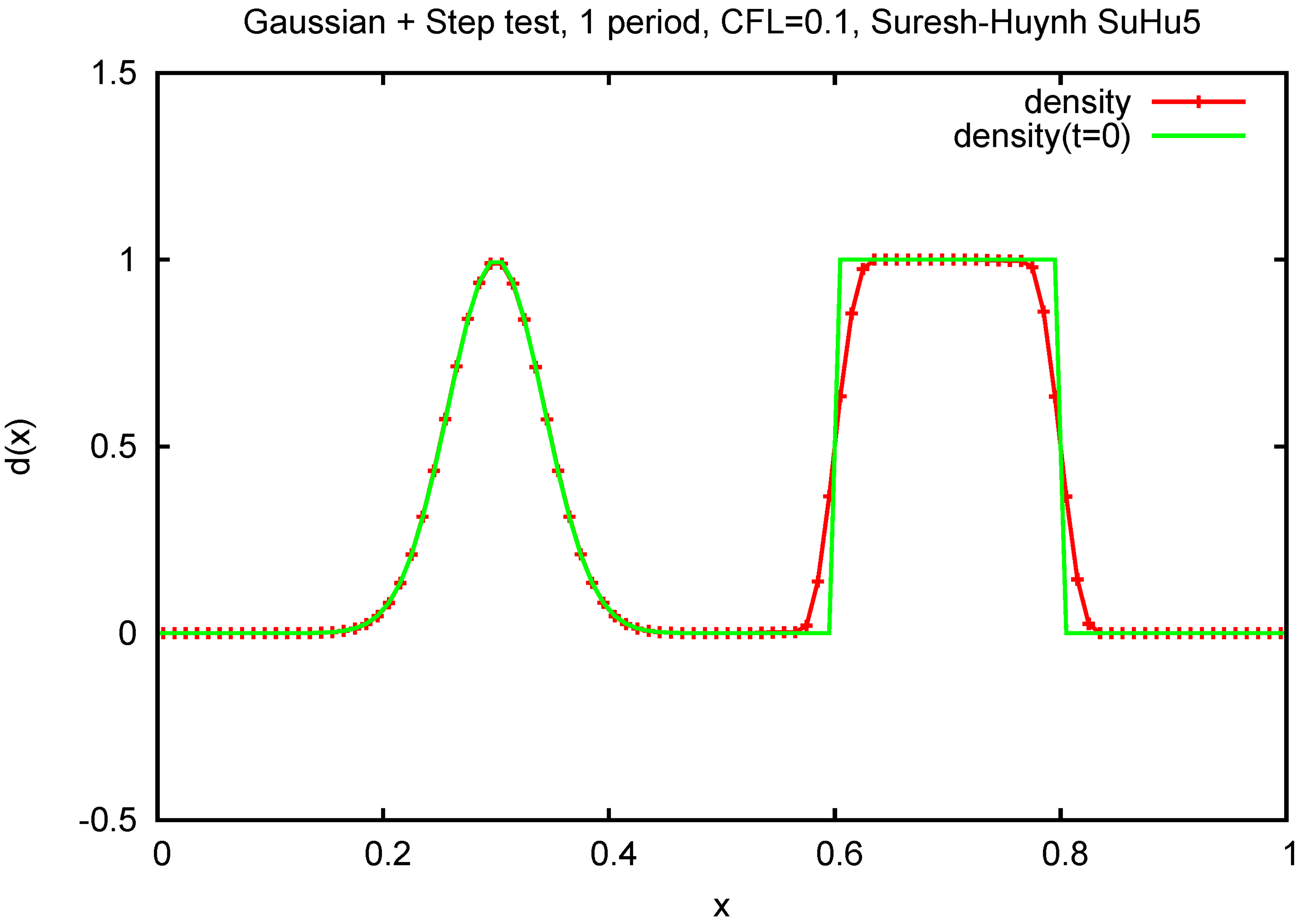
9th order polynomial fit to step function
at 10 discrete points:



bad idea to use very high order polynomial fits: wild oscillations (Gibbs phenomena)

Basis problem: calculus concepts and theorems about convergence apply to sufficiently smooth, well-resolved, functions, and thus break near discontinuities.

State of the art



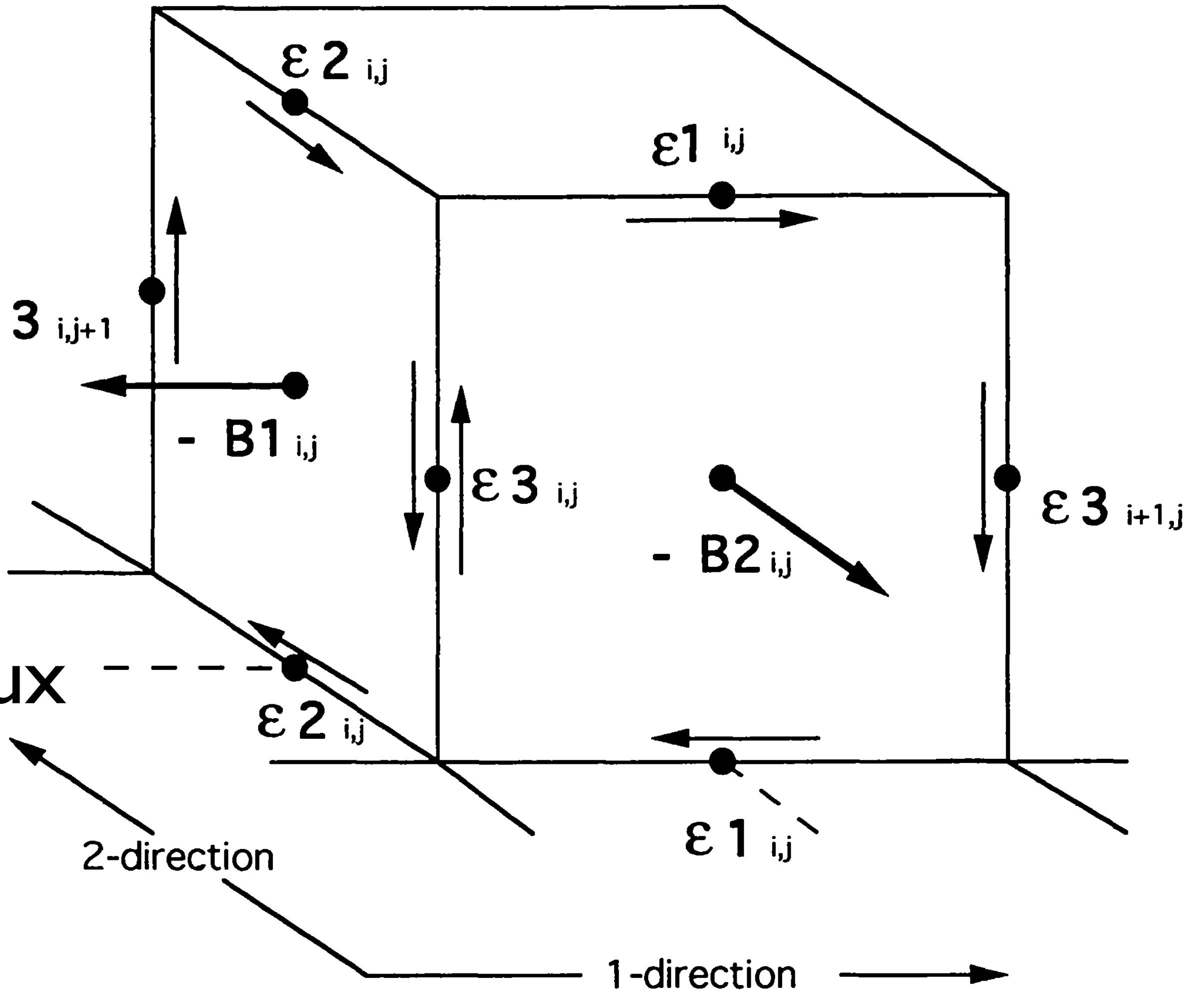
div(B)=0 constraint

evolve the vector potential but need to take second derivatives to get force

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Constrained Transport (CT): integral form of Faraday's induction eq.

$$\frac{\partial \Phi_S}{\partial t} = \oint_{\partial S} (\boldsymbol{v} \times \boldsymbol{B}) \cdot dl$$



Magnetic flux lost through the edge equals flux gained by adjacent cell

Global flux through all faces is conserved, just as mass is conserved in FV approach

Several approaches for div(B)=0

eight-wave formulation: evolution of div(B) terms

projection scheme: solve a Poisson eq. to set div(B)=0, just as is done in spectral schemes for incompressible flows

constrained transport/central difference

Toth 2000 has compared different schemes
Its a very important issue, as improper treatment can give magnetic monopoles & magnetic force along B, etc.

Conclusions

- buoyancy instabilities due to anisotropic conduction
- sophisticated framework for solving HD/ MHD
- several codes available: with various options
- good to have a feel for algorithms: finite volume, REA, CT,....