Methods of Mathematical Physics-I

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Methods of Mathematical Physics I (2021)

- 21 lectures - Mon, Wed, Fri 10 AM - 11 AM

- October : 25, 27, 29

- November : 01, 03, 08, 10, 12, 15, 19, 22, 24, 26, 29

- December : 01, 03, 06, 08, 10, 13

- Mid-term evaluation: 17 November [30%]

- Final evaluation: 17 December [40%]

- Assignment 1: issue 01 Nov; due 08 Nov. [15%]

- Assignment 2: issue 29 Nov; due 06 Dec. [15%]

- Textbooks:
  - Mathematical Methods for Physicists - G.B. Arfken, H.J. Weber & F.E. Harris, Academic Press (Elsevier), 2012
  - Mathematical Methods of Physics - J. Mathews and R.L. Walker, Addison Wesley, 1970
  - The Fourier Transform and its applications - R.N. Bracewell, McGraw-Hill, 2000
  - Data Analysis: A Bayesian Tutorial - D.S. Sivia & J. Skilling, Oxford University Press, 2006

http://www.iucaa.in/~dipankar/mmp1-2021/

Ordinary Differential Equations

x : independent variable

y : dependent variable (unknown): $y(x)$

Linear differential equation: $\mathcal{L}y = f(x)$

$\mathcal{L}$ is a differential operator such that:

$$\mathcal{L}(y_1 + y_2) = \mathcal{L}y_1 + \mathcal{L}y_2 \; ; \; \mathcal{L}(ay) = a\mathcal{L}y \; (a \text{ is a scalar})$$

$$\mathcal{L} = \xi_0(x) + \sum_{k=1}^{n} \xi_k(x)\frac{d^k}{dx^k}$$
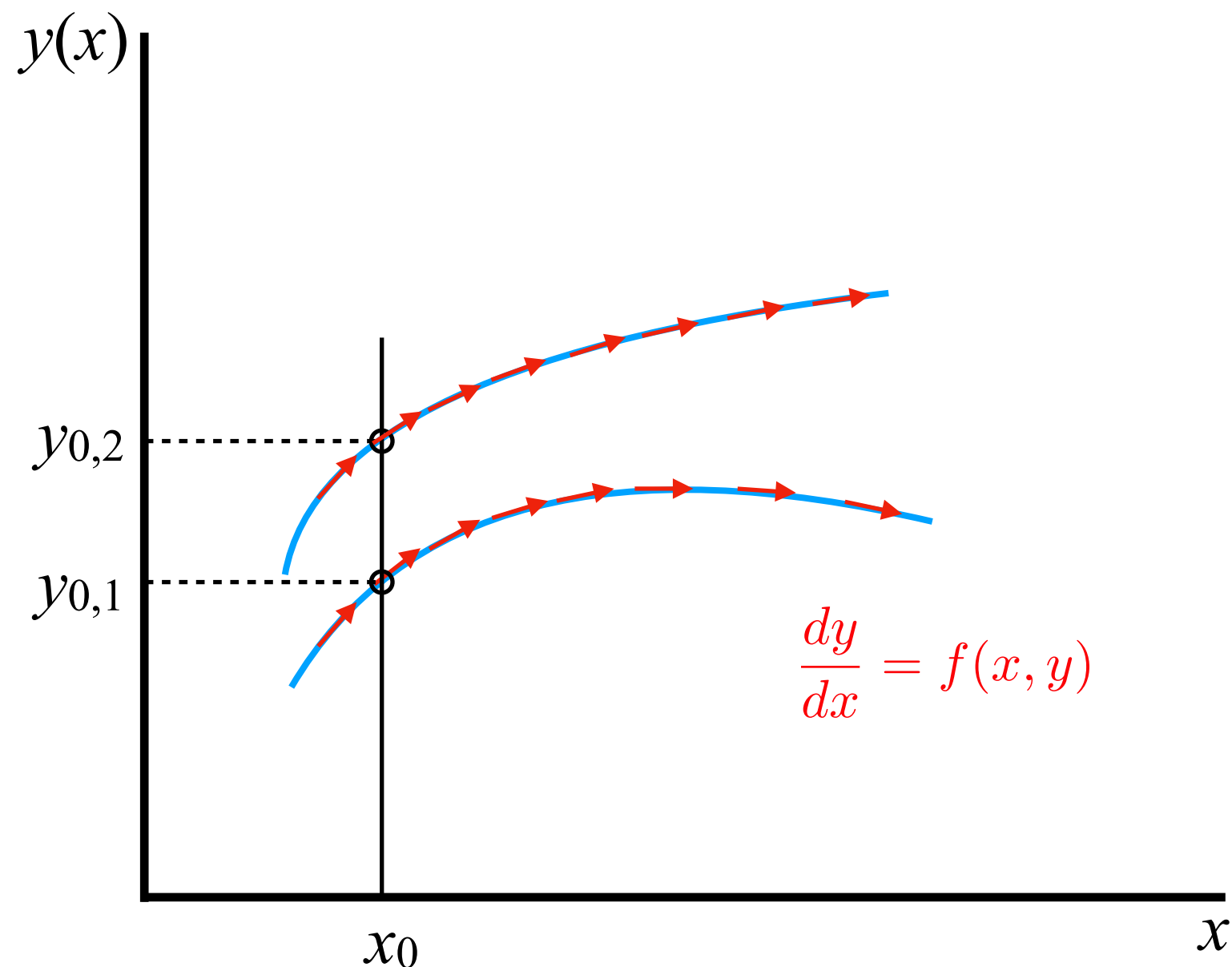
$f(x) = 0$: Homogeneous $f(x) \neq 0$: Inhomogeneous

A differential equation generates a family of solutions

Boundary conditions are required to pick out a specific one

Number of boundary conditions = order of the differential equation

$$\frac{dy}{dx} = f(x, y)$$

First order ODE

$$\frac{dy}{dx} + p(x)y = q(x)$$

The LHS can be converted to a perfect differential by introducing an integrating factor $\alpha(x)$

$$\alpha(x)\frac{dy}{dx} + \alpha(x)p(x)y = \alpha(x)q(x)$$

Such that LHS = $\dfrac{d}{dx}\{\alpha(x)y\}$ $= \alpha(x)q(x)$

which requires $\dfrac{d\alpha(x)}{dx} = \alpha(x)p(x)$, i.e. $\alpha(x) = e^{\int p(x)dx}$

Integration can then evaluate $\alpha(x)y(x)$, subject to B.C.

Pfaffian Forms

$$\mathbf{d}\phi = X(x,y)dx + Y(x,y)dy$$

is a Pfaffian differential form in two variables

Such a form can be defined for any number of independent variables

$\mathbf{d}\phi$ need not be a perfect differential, but if it happens to be then

$$X = \frac{\partial \phi}{\partial x} \quad ; \quad Y = \frac{\partial \phi}{\partial y} \quad \text{and} \quad \frac{\partial X}{\partial y} = \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial Y}{\partial x}$$

If this is not satisfied then $\mathbf{d}\phi$ is not a perfect differential and

$$\oint \mathbf{d}\phi \neq 0$$

Example: in thermodynamics

$$\mathbf{d}Q = dU + pdV$$

A Pfaffian in two variables always admits an integrating factor

Consider $\mathbf{d}\phi = X(x,y)dx + Y(x,y)dy = 0$

Then $\dfrac{dy}{dx} = -\dfrac{X(x,y)}{Y(x,y)}$ is well defined at each (x,y) and can thus

be solved to yield a one-parameter family of curves $\sigma(x,y) = C$

for which $\dfrac{\partial\sigma}{\partial x} + \dfrac{\partial\sigma}{\partial y}\dfrac{dy}{dx} = 0$ and hence $\dfrac{dy}{dx} = -\dfrac{\partial\sigma/\partial x}{\partial\sigma/\partial y}$

so that $X = \tau(x,y)\dfrac{\partial\sigma}{\partial x}$, $Y = \tau(x,y)\dfrac{\partial\sigma}{\partial y}$

and $\mathbf{d}\phi = \tau(x,y)\{\dfrac{\partial\sigma}{\partial x}dx + \dfrac{\partial\sigma}{\partial y}dy\} = \tau(x,y)d\sigma$

thus $\dfrac{\mathbf{d}\phi}{\tau(x,y)} = d\sigma$ is a perfect differential

Integrating denominator

* Not guaranteed in more than two variables

If an integrating factor is admitted then non-intersecting hypersurfaces

$$\sigma(x, y, ...) = \text{constant}$$

can be constructed to represent $\mathbf{d}\phi = 0$

A solution is confined to a hypersurface; will not reach points in the vicinity but not on the surface

In thermodynamics this is ensured by the second law:

Arbitrarily near any given state of a system with any number of thermodynamic variables, there exist states which cannot be reached by reversible adiabatic processes. — C. Caratheodory (1909)

This implies that $\mathbf{d}Q$ admits an integrating denominator:

$$\frac{\mathbf{d}Q}{T} = dS$$

and provides a mathematical definition of Temperature and Entropy

Reference: An introduction to the study of stellar structure - S. Chandrasekhar (1939)

Partial Differential Equations of first order

A. Homogeneous $\qquad \mathcal{L}\phi = a\dfrac{\partial\phi}{\partial x} + b\dfrac{\partial\phi}{\partial y} = 0$ $\qquad$ To solve, need to reduce to ODEs

Method of characteristics

Introduce new variables $\quad s = ax + by \;$ and $\; t = bx - ay \quad$ (orthogonal)

Then $\left(\dfrac{\partial\phi}{\partial x}\right)_y = \left(\dfrac{\partial\phi}{\partial s}\right)\left(\dfrac{\partial s}{\partial x}\right)_y + \left(\dfrac{\partial\phi}{\partial t}\right)\left(\dfrac{\partial t}{\partial x}\right)_y = a\left(\dfrac{\partial\phi}{\partial s}\right) + b\left(\dfrac{\partial\phi}{\partial t}\right)$

Similarly $\quad \left(\dfrac{\partial\phi}{\partial y}\right)_x = b\left(\dfrac{\partial\phi}{\partial s}\right) - a\left(\dfrac{\partial\phi}{\partial t}\right)$
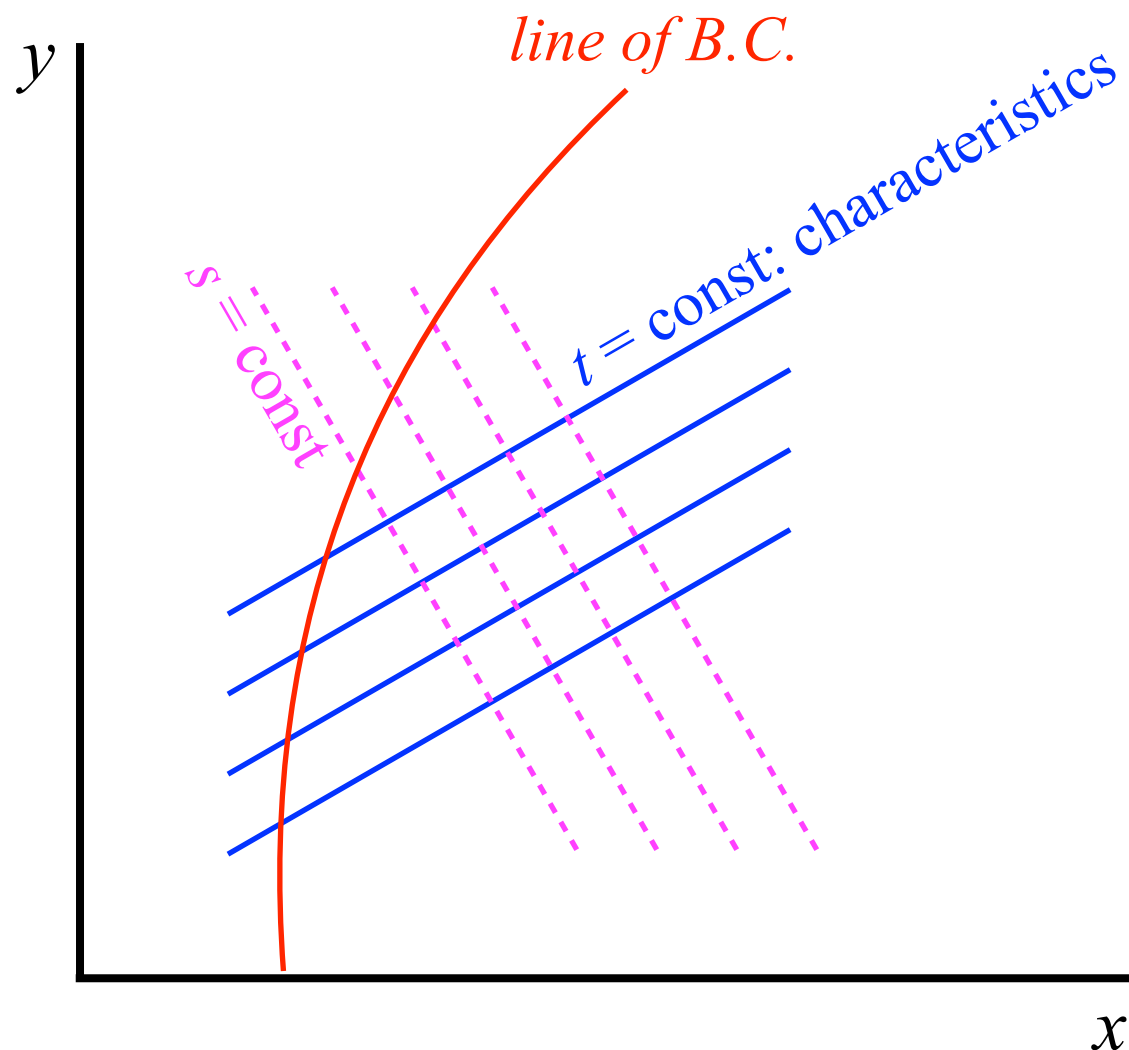
So

$$\mathcal{L}\phi = a^2\left(\dfrac{\partial\phi}{\partial s}\right) + ab\;\cancel{\left(\dfrac{\partial\phi}{\partial t}\right)} + b^2\left(\dfrac{\partial\phi}{\partial s}\right) - ab\;\cancel{\left(\dfrac{\partial\phi}{\partial t}\right)} = 0$$

$$\mathcal{L}\phi = (a^2 + b^2)\frac{\partial \phi}{\partial s} = 0 : \ \phi \text{ is independent of } s : \ \phi(s,t) = \phi(t)$$

$\phi(x,y)$ is an arbitrary function of $(bx - ay)$

To be specified by boundary conditions

$\phi(x,y) =$ constant   on lines of $(bx - ay) =$ constant : *Characteristics*

Specify boundary conditions on a line.

Propagate values along characteristics from intersection points, to fill the full plane.

Boundary curve crossing a characteristic more than once may lead to inconsistency.

Boundary curve coincident with a characteristic will be either irrelevant or inconsistent.

Partial Differential Equations of first order

B. Inhomogeneous $\quad \mathcal{L}\phi = a\dfrac{\partial \phi}{\partial x} + b\dfrac{\partial \phi}{\partial y} + q(x,y)\phi = F(x,y)$

Use the characteristics of $\quad a\dfrac{\partial \phi}{\partial x} + b\dfrac{\partial \phi}{\partial y} = 0 : \qquad x = \dfrac{as+bt}{a^2+b^2}, \; y = \dfrac{bs-at}{a^2+b^2}$

$$\Rightarrow \qquad (a^2+b^2)\dfrac{\partial \phi}{\partial s} + \hat{q}(s,t)\phi = \hat{F}(s,t)$$

This is now an ODE with t as a parameter. The ODE in s may be solved using usual techniques.

For a PDE of the form $\quad a(x,y)\dfrac{\partial \phi}{\partial x} + b(x,y)\dfrac{\partial \phi}{\partial y} = C(x,y)$

We note $\quad \dfrac{dx}{a(x,y)} = \dfrac{dy}{b(x,y)} = \dfrac{d\phi}{C(x,y)}$

Characteristics may be found by solving $\qquad \dfrac{dx}{ds} = a(x,y); \; \dfrac{dy}{ds} = b(x,y)$

and eliminating s from the result.

Partial Differential Equations of first order

In 3-D:
$$a\frac{\partial \phi}{\partial x} + b\frac{\partial \phi}{\partial y} + c\frac{\partial \phi}{\partial z} = 0$$

Define $s = ax + by + cz$ and two other coordinates t and u such that (s, t, u) form an orthogonal triad

This will reduce the equation to $(a^2 + b^2 + c^2)\frac{\partial \phi}{\partial s} = 0$

Hence the solutions are $\phi(s, t, u) = f(t, u)$

On lines with fixed t and u, the solution is constant for all s

These are the *characteristics*

Boundary conditions are to be specified on surfaces
- Not coincident with characteristics
- Not intersecting a characteristic more than once