Methods of Mathematical Physics-I

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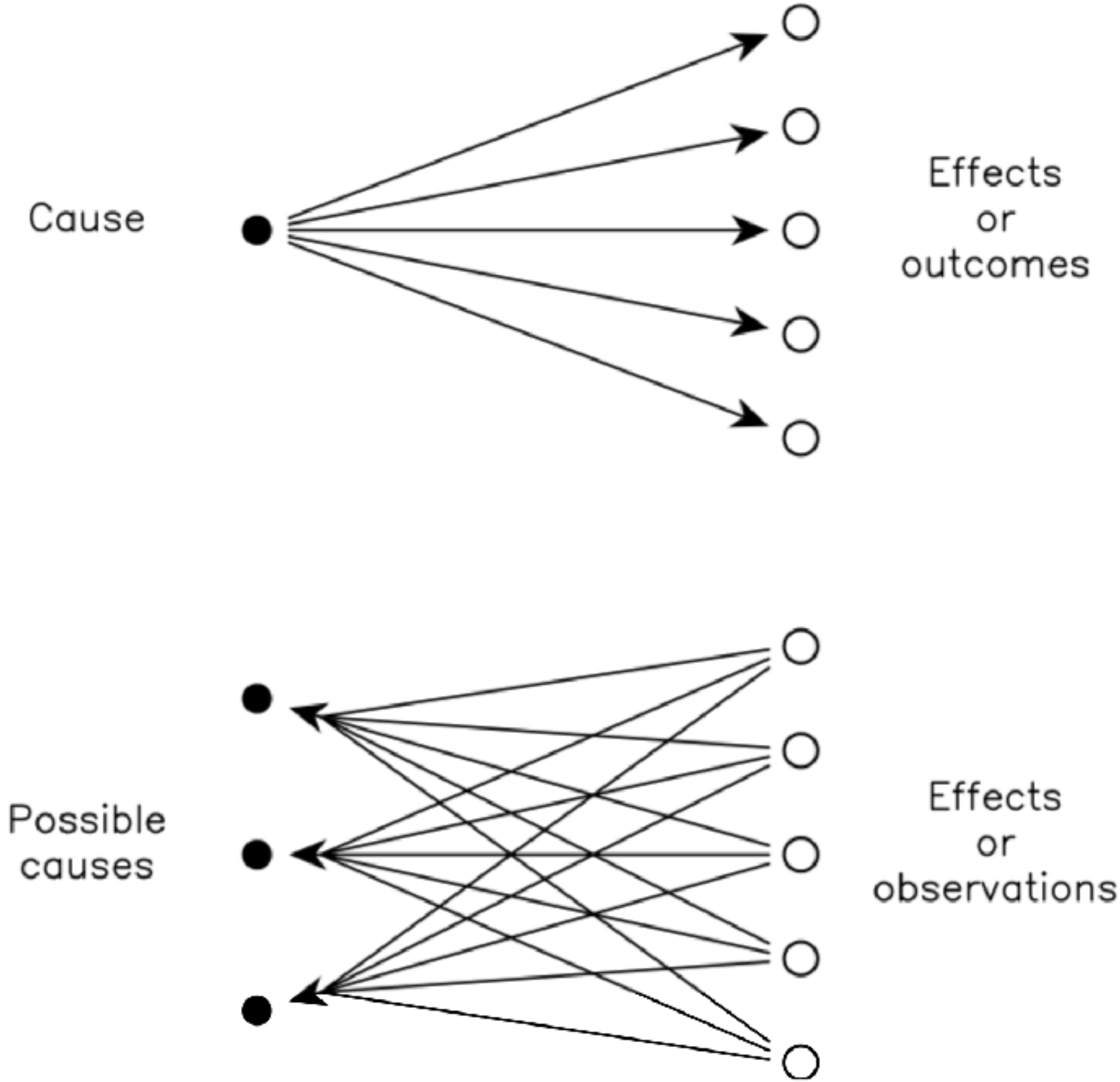
Bayesian Inference



Deductive *vs* Inductive logic.

The latter is often used in Scientific inference.

Sivia and Skilling 2006

All probabilities are conditional: $p(A|B)$ - depends on prior B

Let

Available prior information: I

Data resulting from experiment: D

Hypothesis for a cause: H

$p(X|I)$ represents the probability that X is true, given I (information)

$p(\bar{X}|I)$ represents the probability that X is false, given I (information)

$$p(X|I) + p(\bar{X}|I) = 1$$

One may write $p(X, Y|I) = p(X|Y, I) \times p(Y|I)$

Also $p(X, Y|I) = p(Y|X, I) \times p(X|I)$

Therefore $p(X|Y, I)p(Y|I) = p(Y|X, I)p(X|I)$

Thus
$$p(X|Y, I) = \frac{p(Y|X, I)p(X|I)}{p(Y|I)}$$

This result is known as the Bayes' Theorem [Bayes 1763]

In the above, *X* may be replaced by Hypothesis *H* and *Y* by Data *D*. Then

$$p(H|D, I) = \frac{p(D|H, I)p(H|I)}{p(D|I)}$$

The LHS here is called the *Posterior*, and on the RHS $p(H|I)$ is called the *Prior*. $p(D|H, I)$ is known as the *Likelihood* and $p(D|I)$ the *Evidence*.

We also note

$$p(X, Y|I) = p(Y, X|I) = p(Y|X, I) \times p(X|I)$$

$$p(X, \bar{Y}|I) = p(\bar{Y}, X|I) = p(\bar{Y}|X, I) \times p(X|I)$$

So $$p(X, Y|I) + p(X, \bar{Y}|I) = [p(Y|X, I) + p(\bar{Y}|X, I)] \times p(X|I)$$

i.e. $$p(X|I) = p(X, Y|I) + p(X, \bar{Y}|I)$$

The two terms on the RHS together encompass the exhaustive set of possibilities for *Y*.

Consider that *Y* takes a discrete set of possible values $\{Y_k\} = Y_1, Y_2, Y_3 ...., Y_M$

Then $p(X|I) = \sum_{k=1}^{M} p(X, Y_k|I)$, subject to $\sum_{k=1}^{M} p(Y_k|I) = 1$

If *Y* is a continuous variable (a *parameter*), then

$$p(X|I) = \int_{-\infty}^{\infty} p(X, Y|I)\, dY$$ This is called *Marginalisation*

Examples of Bayesian Inference

Game Show problem:

- There are three closed doors, there is a car behind one of them
- The contestant wins the car if the correct door is chosen
- Contestant chooses door 1
- The host opens door 3 and shows that it has no car behind it
- Contestant is asked whether to stick with original choice or choose door 2

What should the contestant do to maximise the probability of winning?

Hypotheses H_n : car is behind door n

Prior: $p(H_1|I) = p(H_2|I) = p(H_3|I) = \dfrac{1}{3}$

Data: Door 3 was opened when Door 1 was chosen. Car is not behind Door 3.

We note then $p(D|H_3, I) = 0, \quad p(D|H_1, I) = \dfrac{1}{2}, \quad p(D|H_2, I) = 1$

So $p(D|I) = \sum_n p(D|H_n, I)p(H_n|I) = \frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} = \frac{1}{2}$

$$\therefore \quad p(H_1|D,I) = \frac{p(D|H_1,I)p(H_1|I)}{p(D|I)} = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}$$

$$p(H_2|D,I) = \frac{p(D|H_2,I)p(H_2|I)}{p(D|I)} = \frac{1 \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

Thus changing the choice to Door 2 will double the chances of winning the car.

Medical test for a rare disease

- Say one in 10^4 people have such a disease
- There is a good test available, with false positivity rate of 2% and false negativity rate of 1%
- If a patient tests positive in this test then what is the probability that the patient really has this disease?

Hypothesis H: patient has the disease; $\bar{H}$: patient does not have the disease

Prior: $$p(H|I) = 10^{-4}; \quad p(\bar{H}|I) = 1 - 10^{-4}$$

Data D: patient tests positive:

$$p(D|H,I) = 1 - 0.01 = 0.99; \quad p(D|\bar{H},I) = 0.02$$

So $$p(D|I) = p(D|H,I)p(H|I) + p(D|\bar{H},I)p(\bar{H}|I)$$

$$= 0.99 \times 10^{-4} + 0.02 \times (1 - 10^{-4}) = 0.020097$$

And $$p(H|D,I) = \frac{p(D|H,I)p(H|I)}{p(D|I)} = \frac{0.99 \times 10^{-4}}{0.020097} = 4.93 \times 10^{-3}$$

= 0.493%