Methods of Mathematical Physics-I

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$$\frac{\mathrm{e}^{ilx}}{\sqrt{2\pi}} \quad \text{or} \quad \frac{\sin(lx)}{\sqrt{\pi}}, \frac{\cos(lx)}{\sqrt{\pi}} \quad \begin{array}{l}\text{are orthonormal basis functions}\\ \text{for } x \text{ in } [0, 2\pi]\end{array}$$

Any function defined in the interval $[0, 2\pi]$ may therefore be expanded in this basis. This is true of all periodic functions. x may be scaled to define the period as 2π and the interval $[0, 2\pi]$ may be shifted by any amount with the length 2π kept constant.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \quad \text{or} \quad f(x) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{inx}$$

where

$$c_n = \frac{1}{2}(a_n - ib_n); \quad c_{-n} = \frac{1}{2}(a_n + ib_n); \quad c_0 = \frac{1}{2}a_0; \quad (n > 0)$$

and

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(s) \cos(ns) \, ds; \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(s) \sin(ns) \, ds;$$

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(s) \mathrm{e}^{-ins} \, ds$$

Fourier Series

The Fourier series definition is valid over all space $(-\infty < x < \infty)$ for any periodic function, with the function and the corresponding expansion repeating at the intervals of 2π.

▸ The first term is the average of the function in the interval $[0, 2\pi]$: "DC component"

▸ For a zero-mean function $a_0 = c_0 = 0$

▸ a_n terms contribute the even part of the series and b_n terms the odd part

▸ If $f(x)$ has a discontinuity in the domain then at the point of discontinuity the Fourier series evaluates to the arithmetic average of the left and the right hand limits:

$$f_{\text{Fourier}}(x_0) = \lim_{\epsilon \to 0} \left[\frac{f(x_0 + \epsilon) + f(x_0 - \epsilon)}{2} \right]$$

Dirichlet conditions on $f(x)$:

▸ Finite number of finite discontinuities in the defined interval $[0, 2\pi]$

▸ Finite number of extreme values in the defined interval

If the function is an oscillation with a period P_0, then $x = \omega_0 t; \quad \omega_0 = 2\pi/P_0$

Function is periodic in x (phase), defined over $[0, 2\pi]$.

So
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

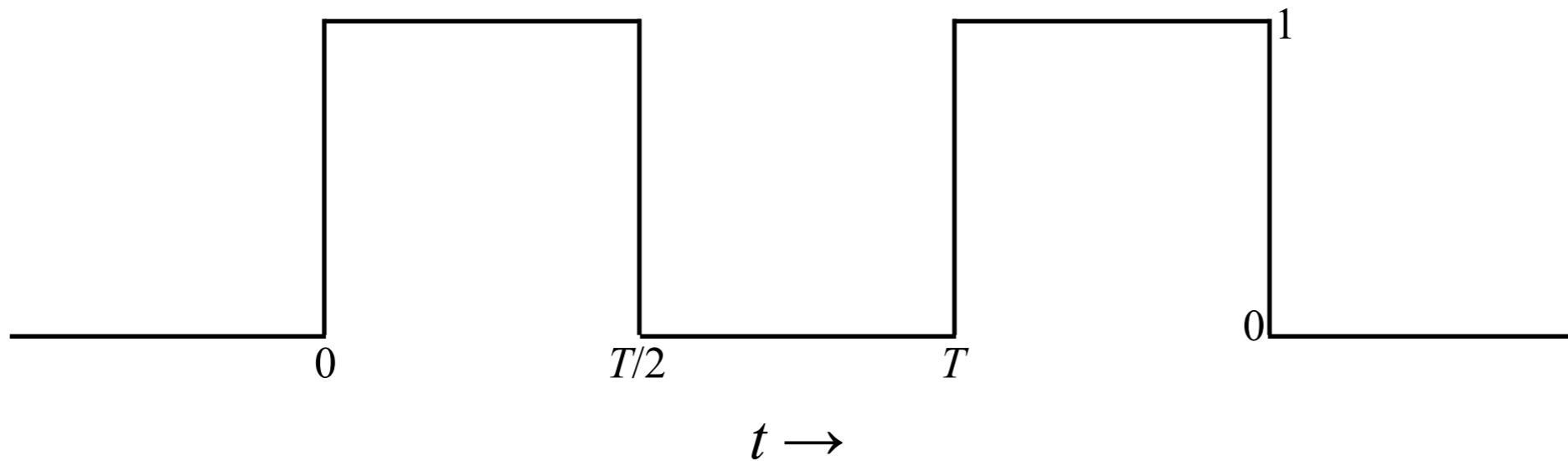$$= \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t} = \sum_{n=0}^{\infty} \beta_n e^{i(n\omega_0 t - \phi_n)}$$

For travelling waves, periodicity is at $x = \lambda$, the wavelength.

Defining $k_0 = \dfrac{2\pi}{\lambda}$ (wavenumber), $k_0 x$ has the range $[0, 2\pi]$.

i.e.
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nk_0 x) + b_n \sin(nk_0 x)]$$

$$= \sum_{n=0}^{\infty} \beta_n e^{i(nk_0 x - \phi_n)}$$

Example: Square wave oscillation of period T

$$f(t) = 1 \quad (0 \le t < T/2)$$
$$= 0 \quad (T/2 \le t < T)$$

$$\omega_0 = \frac{2\pi}{T}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right]$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(t) \, d(\omega_0 t) = \frac{\omega_0}{\pi} \int_0^T f(t) \, dt = \frac{2}{T} \cdot \frac{T}{2} = 1$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos(n\omega_0 t) \, d(\omega_0 t) = \frac{2}{T} \int_0^T f(t) \cos(n\omega_0 t) \, dt$$

$$= \frac{2}{T} \int_0^{T/2} \cos(n\omega_0 t) \, dt = \frac{2}{T} \left. \frac{\sin(n\omega_0 t)}{n\omega_0} \right|_0^{T/2} = \frac{1}{n\pi} [\sin(n\pi) - \sin(0)] = 0$$

$$b_n = \frac{1}{n\pi} \cos(n\omega_0 t)|_{T/2}^0 = -\frac{1}{n\pi} [\cos(n\pi) - \cos(0)] = \frac{2}{n\pi} \quad \text{(odd } n)$$

$$= 0 \quad \text{(even } n)$$

Thus

$$f(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \sin\left(n\frac{2\pi}{T}t\right)$$
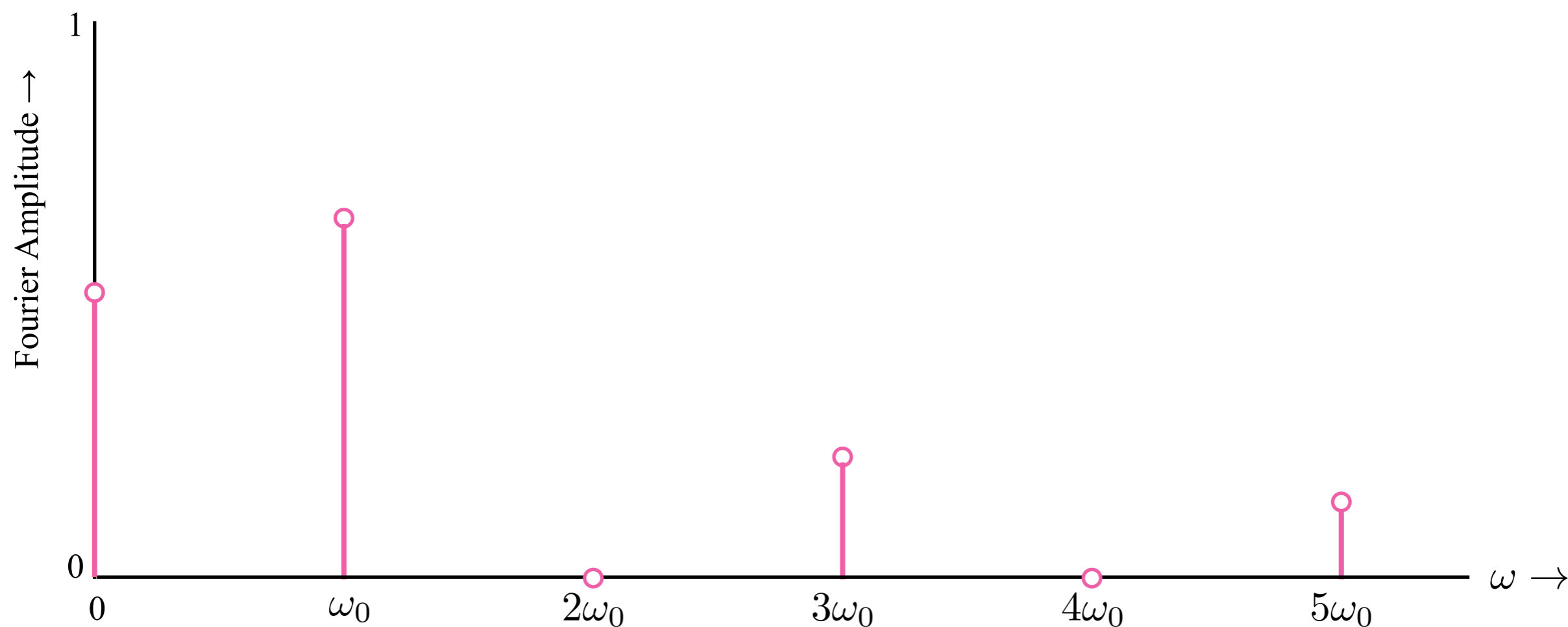
Fourier components consist of

▶ Oscillation at frequency ω_0: *fundamental*

▶ Oscillations at integral multiples of ω_0 : *harmonics* or *overtones*

   In this case the even harmonics are absent

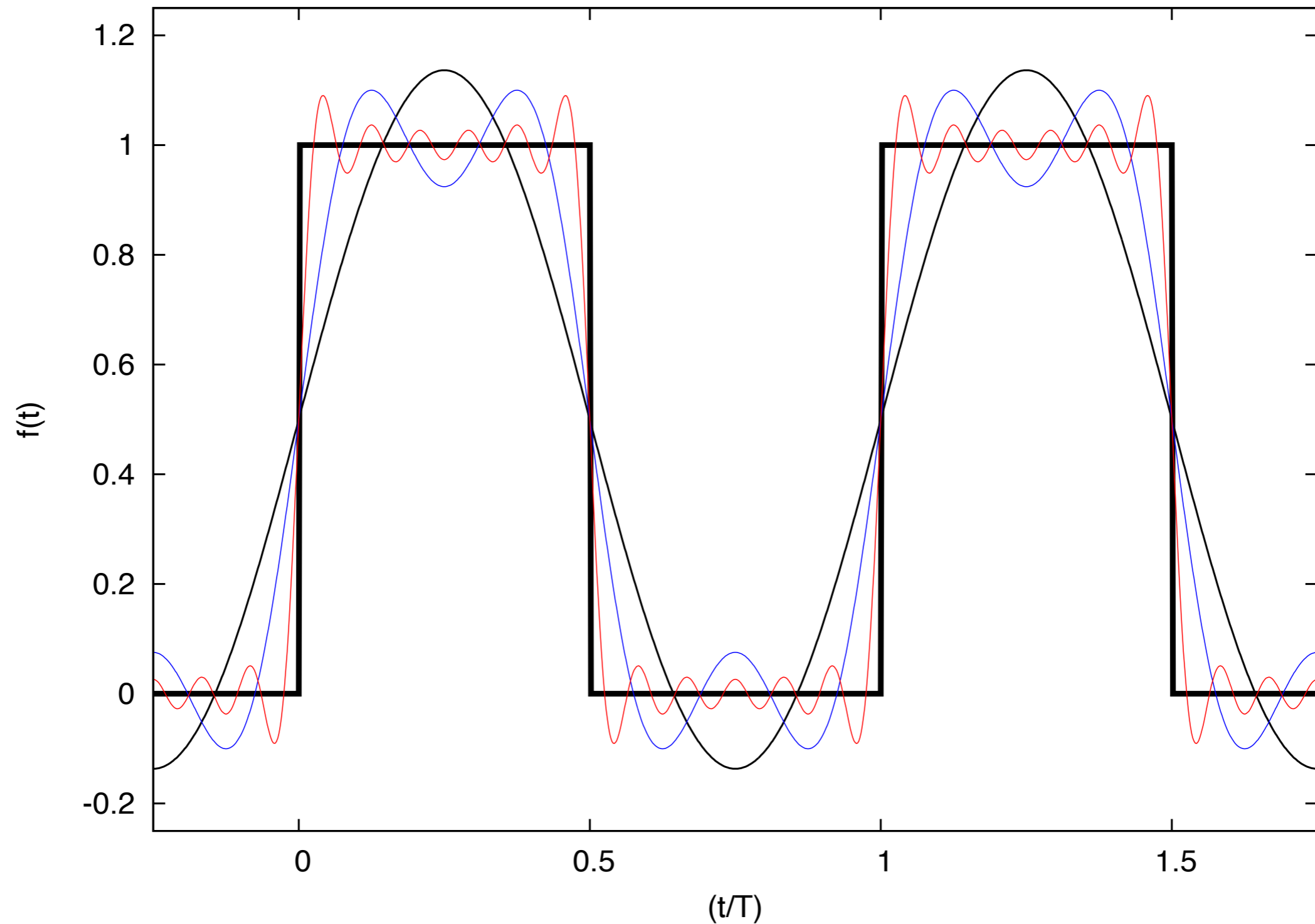▶ A *DC component* : 1/2 : equal to the average over the period

   If the function were zero mean: $\left[f(t) - \dfrac{1}{2}\right]$ then no DC would be present

Graphical representation of the Fourier components

This is a discrete "spectrum"

Fourier series decomposition of a square wave

Thick black line: original function   Blue line: sum up to harmonic no. 3 (*i.e.* three non zero terms)

Thin black line: DC+fundamental   Red line: sum up to harmonic no. 11 (*i.e.* seven non zero terms)