Plasma Oscillations

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Abstract

Plasma Oscillations were first observed in 1929, in relation to the large fluctuations in the velocities of electrons in the low pressure mercury arc. Plasma Oscillation is an example of the collective phenomena that can occur in a plasma. It is a fundamental excitation mode that can occur in a plasma. In this article we take a brief review of plasma properties, basic theory of plasma and plasma oscillations.

Contents

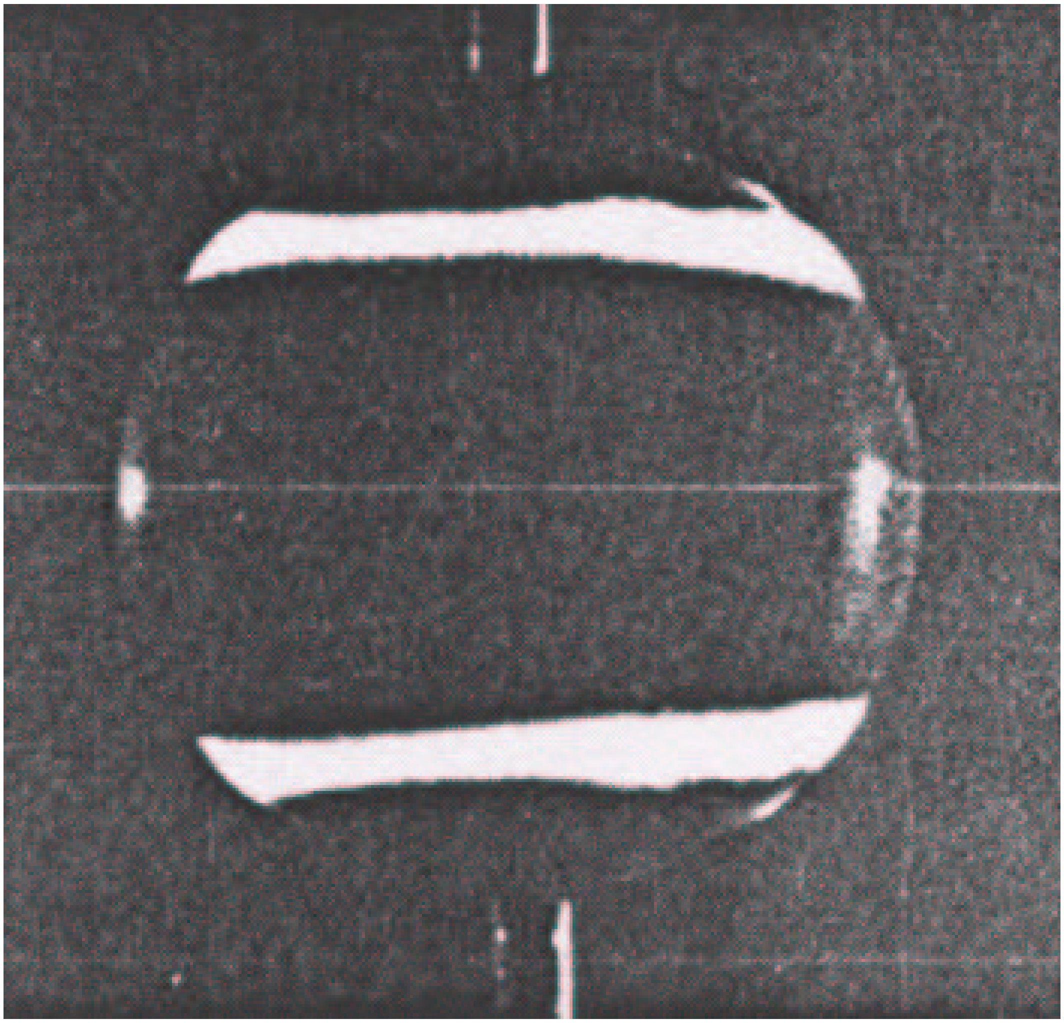

Figure 1: Birkland's terella experiment.

1 Introduction

To begin with, a few words towards the history of the plasma physics are in order. The early classical theory of plasma was based on the knowledge of kinetic theory of gases. These elegant mathematical derivations did not take into account any laboratory level experiments, but were applied directly to cosmic plasma. The disagreements with the experimental plasma were discarded as too complicated. The first attempt to connect cosmic plasma and laboratory plasma physics was done by Birkeland in 1908. He observed aurorae and magnetic storms in nature and set up experiments to observe them in the lab, as shown in Figure 1. Figure 1 is a photograph of a modern version of his Terella experiments, demonstrating what happens when a magnetised sphere is immersed in a plasma. The luminous rings around the poles were identified as the auroral zones.

Our knowledge of cosmic plasma was also limited to ground based observations. So, e.g. the magnetic field mapping around the earth changed significantly with the advent of space missions, as shown in Figure 2. As more and more data from "in situ" observations became available, there were more and more discrepancies from the predicted behaviour from classical plasma theory. This was because there were more and more terms in the equations, which became significant on the different scales of length. The scaling from the laboratory plasma to astrophysical plasma is difficult because of different scaling laws of various plasma parameters.

Some examples of natural and man-made plasma are listed below:

- *Laboratory gas discharge.* The plasmas created in laboratory by electric currents flowing through hot gas, e.g., in vacuum tubes, spark gaps, welding arcs, and neon nd fluorescent lights.
- *Controlled thermonuclear fusion experiments.* The plasmas in which experiments for controlled thermonuclear fusion are carried out, e.g., in tokamaks.
- *Ionosphere.* The part of the earth's upper atmospher, (at heights of $\sim$ 50 - 300 km), that is partially photoionised by solar ultravoilet radiation.
- *Magnetosphere.* The plasma of high-speed electrons and ions that are locked onto the earth's dipoar magnetic field and slide around on its field lines at several earth radii.
- *Sun's core.* The plasma at the center of the sun, where fusion of hydrogen to form helium generates the sun's heat.
- *Solar wind.* The wind of plasma that blows off the sun and outward through the region between the planets.
- *Interstellar medium.* The plasma, in our Galaxy, that fills the region between the stars; this plasma exhibits a fairly wide range of density and temperature as a result o such processes as heating by photons from stars, heating and compression by shock waves from supernovae, and cooling by thermal emission of radiation.
- *Intergalactic medium.* The plasma that fills the space outside galaxies and clusters of galaxies.

The characteristic values of the various plasma parameters of these systems are listed in Table 1. The definitions of these parameters follow in the text below.

The degree of Ionisation x of these plasma can be obtained from the Saha equation, as given by

$$\frac{x^2}{x-1} = \frac{(2\pi m_e)^2}{h^3}\frac{\kappa_B T}{p_{gas}} exp(-\frac{\chi}{\kappa_B T}) \tag{1}$$

where χ is the ionisation energy. This equation holds for plasmas which are in thermal equilibrium. The regions of ionized hydrogen around the early-type stars known as H II regions are almost completely ionized by the ultraviolet radiation coming from the central stars and not in thermodynamic equilibrium. To determine their ionisation,

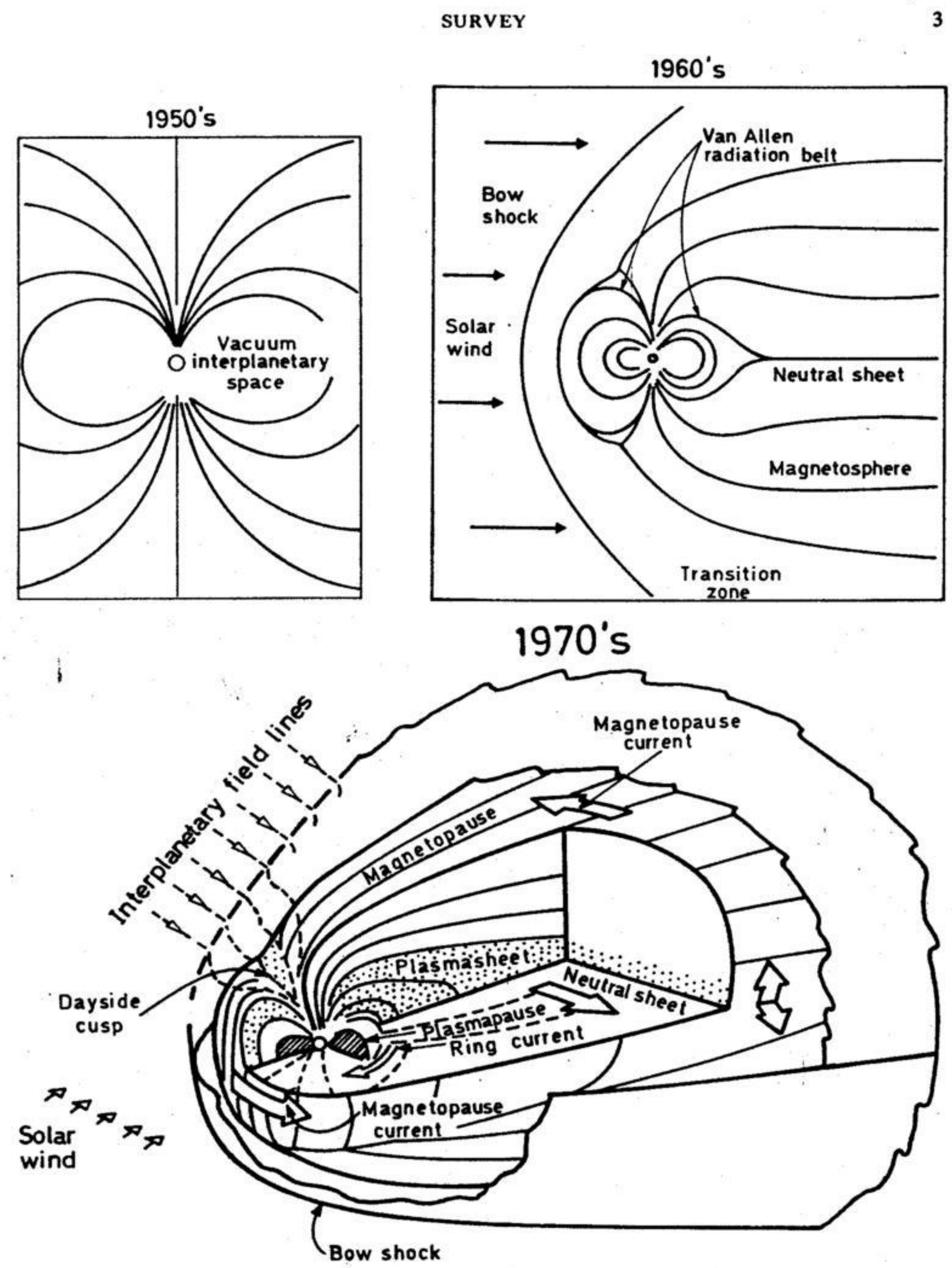

Figure 2: (a)Up to the beginning of the space age it was generally assumed that the Earth was surrounded by vacuum and its dipole field was unperturbed(except at magnetic storms). (b) The first space measurements showed the existence of the VAn Allen belts, the magnetopause, the neutral sheet in the tail and the bow shock. (c) New measurements have made the magnetic field description increasingly complicated.

Table 1: Plasma parameters

Plasma	$n_e(m^{-3})$	T (K)	B (T)	$\lambda_D(m)$	$\omega_p(s^{-1})$
Gas discharge	10^{16}	10^4	-	10^{-4}	10^{10}
Tokamak	10^{20}	10^8	10	10^{-4}	10^{12}
Ionosphere	10^{12}	10^3	10^{-5}	10^{-3}	10^8
Magnetosphere	10^7	10^7	10^{-8}	10^2	10^5
Solar core	10^{32}	10^7	-	10^{-11}	10^{18}
Solar wind	10^6	10^5	10^{-9}	10	10^5
Interstellar Medium	10^5	10^4	10^{-10}	10	10^4
Intergalactic Medium	1	10^6	-	10^5	10^2

we use the flux of uv photons from the central stars. It is estimated that 99 % of the material in the Universe exists in the plasma state, even though Saha eqn may not be applicable to some of this plasma.

2 Fundamentals of Plasma

Plasma is a fluid which contains ions and electrons, such that overall charge neutrality is maintained. Simple examples include a gas heated upto sufficiently high temperatures so that the atoms ionise. Another example is that of a liquid Sodium.

If we consider a fluid element of plasma, it is overall charge neutral. So an external electric field cannot cause motion of a fluid element as a whole, but will set up currents, which is motion of opposite charges in opposite directions. Due to these currents, an external Magnetic field can exert force on the fluid element, changing its direction of motion. The study of effects of electromagnetic fields on a neutral conducting medium is called *'Magnetohydrodynamics.'*

One model used to describe motion of plasma is the Two fluid model, in which the positive and negative charges are treated as separate fluids. Let m_e, m_i,q_e, q_i, and n_e, n_i denote the mass, charge, and density of electrons and positive ions respectively.

We assume that an element of fluid is in thermal equilibrium at a temperature T. To see the effects of one charge on the other, let's suppose that we place a test charge of magnitude Q. This charge will create a potential $\phi(r)$ such that $\phi \to 0$ as $r \to \infty$. Suppose this charge induces a charge density $n_e(r) = n_{e0}e^{-q_e\phi/kT}$ and $n_i(r) = n_{i0}e^{-q_i\phi/kT}$. Note that, $n_e(r) \to n_{e0}$ as $r \to \infty$ and $n_i(r) \to n_{i0}$ as $r \to \infty$. Then, charge neutrality requires that $q_e n_{e0} + q_i n_{i0} = 0$.

Poisson's equation for potential distribution of a charge Q and the sea of electrons and ions is

$$\nabla^2\phi = -4\pi\{Q\delta(\vec{r}) + q_e n_e(r) + q_i n_i(r)\} \tag{2}$$

Substituting n_e and n_i, and retaining the terms upto first order in $q_e\phi/kT$, we get

$$\nabla^2\phi \approx -4\pi\{Q\delta(\vec{r}) + q_e n_{e0}(1 - \frac{q_e\phi}{kT}) + q_i n_{i0}(1 - \frac{q_i\phi}{kT})\} \tag{3}$$

$$= -4\pi\{Q\delta(\vec{r}) - (\frac{q_e^2 n_{e0} + q_i^2 n_{i0}}{kT})\phi\} \tag{4}$$

Introducing a parameter λ_D called Debye Length,

$$\lambda_D^2 = \frac{kT}{4\pi(q_e^2 n_{e0} + q_i^2 n_{i0})} \tag{5}$$

we can rewrite the equation as

$$\nabla^2\phi = -4\pi Q\delta(\vec{r}) + \phi/\lambda_D^2 \tag{6}$$

The solution of this equation is

$$\phi(r) = \frac{Q}{r} exp\{-\frac{r}{\lambda_D}\} \tag{7}$$

Note that $\phi \rightarrow \frac{Q}{r}$ as $r \rightarrow 0$. Thus the field remains that of a single charge Q for distances smaller than λ_D. For distances larger than λ_D, the field dies off. This is the screening effect, due to creation of a polarisation cloud around a charge Q, of the charges of opposite signs, which screens the field of charge for distances larger than the Debye length. Thus charge fluctuations in plasma may occur over distances smaller than λ_D. For a plasma to be considered as a neutral fluid, the number of particles in a volume of the size of Debye legth should be much larger than one. i.e. the plasma parameters defined by $g_e \sim n_{e0}\lambda_D^3$ and $g_i \sim n_{i0}\lambda_D^3$ should be much larger than unity.

The equations of Two Fluid Model are simply the equations of electrons and ions as separate fluids. Extra force terms as compared to nonconducting fluids are added on RHS of Euler Equation, due to Electromagnetic fields and interparticle collisions. We can treat plasma as a single fluid under certain assumptions, mainly

- Plasma is quasi neutral, i.e. $\rho_q \sim 0 \Rightarrow n_e \simeq Zn_i$. $L \gg \lambda_D$ and $\tau \gg 1/\omega_p$. For these condition to be true for all times, $\nabla \cdot \vec{J} = 0$
- Drift Velocity of electrons small. i.e. the current density is $\vec{J} \ll n_e eU$ where U is the characteristic velocity in the fluid.
- small m_e. $\Rightarrow m_e \frac{d\vec{V}_e}{dt} \approx 0$
- pressure forces much smaller than electromagnetic forces.

We also introduce the averaged quantities suitable for single fluid as:

$\rho_m = n_i m_i + n_e m_e$

$\rho_q = n_i q_i + n_e q_e$

$\vec{V} = \frac{n_e m_e i\vec{V}_e + n_i m_i \vec{V}_i}{n_i m_i + n_e m_e}$

$\vec{J} = n_e q_e \vec{V}_e + n_i q_i \vec{V}_i$

With these definitions, using generalised ohm's law, we can derive the general equations of Magnetohydrodynamics as follows:

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{V}) = 0 \tag{8}$$

$$\rho(\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V}) = -\nabla P + \frac{\vec{J} \times \vec{B}}{c} \tag{9}$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{V} \times \vec{B}) + \eta \nabla^2 \vec{B} \tag{10}$$

where, $\eta = c^2/4\pi\sigma$ is called magnetic diffusivity.

3 Passage of EM waves through Plasma.

Let's assume that a sinusoidal electric field is incident on a sea of electrons. After writing the force equations for one electron and solving for motion of electrons, we get

$$\epsilon(\omega) = 1 + \frac{4\pi \mathrm{Ne}^2}{\mathrm{m}} \sum_j \frac{\mathrm{f_j}}{\omega_\mathrm{j}^2 - \omega^2 - \mathrm{i}\omega\gamma_\mathrm{j}} \tag{11}$$

If ω is larger than the natural binding frequencies ω_j , then the dielectric constant becomes

$$\epsilon(\omega) = 1 - \frac{\omega_\mathrm{p}^2}{\omega^2} \tag{12}$$

where

$$\omega_p^2 = \frac{4\pi \mathrm{NZe}^2}{\mathrm{m}} \tag{13}$$

is Plasma Frequency. Note that it ω_p depends only on the total number of electrons in a unit volume.

The wave number is given by $ck = \sqrt{\omega^2 - \omega_p^2}$

And the dispersion relation is then:

$$\omega^2 = \omega_p^2 + c^2 k^2 \tag{14}$$

For k to be real, only those electromagnetic waves are allowed to pass, for which $\omega > \omega_p$.

At very high frequencies, $\omega = ck$, thus electrons can't respond fast enough, and plasma effects are negligible.

Thus Plasma frequency sets the lower cutoff for the frequencies of electromagnetic radiation that can pass through a plasma. The metals shine by reflecting most of light in visible range. The visible light can't pass through the metal because the plasma frequency of electrons in metal falls in ultraviolet region. For frequencies in UV, metals are transparant.

The earth's ionosphere reflects Radio waves for the same reason, though the analysis is not so straightforward. The electron densities at various heights in the ionosphere can be inffered by studying the reflection of pulses of radiation transmitted vertically upwards. Also, the broadcast of various Radio signals in communication on Earth is possible only because of reflection from the ionosphere.

4 Plasma Oscillations

The story of these plasma oscillations begins with Langmuir's observations in low pressure mercury vapur discharge tube. He observed that under a wide range of conditions, there were many electrons with abnormally large velocities, whose voltage equivalent is greater than the total voltage drop across the tube. There was an even larger number of electrons with Kinetic energies lower than the average KE, so the group as a whole has not acquired extra energy, but there has been a redistribution of energy. One such mechanism for such rapid transfer of energy between the electrons was suggested as scattering of electrons due to rapidly changing electronic fields.Dittmer obtained evidence pointing in this direction and Penning observed such oscillations of radio frequencies in low pressure mercury and argon vapor discharges. Finally, Tonks and Langmuir came up with a simple theory and experimental observations of these oscillations.

Now coming to back to our main topic, we will consider two main interactions, that of radiation with plasma and that of a beam of electrons with the plasma. When radiation of frequency ω is incident on a plasma, three modes of oscillation can be supported. Two are transverse and one longitudinal. We assume that the electrons are so mobile that the motion of ions is negligible and the electrons are embeded in a sea of immobile ions.

4.1 Plasma Electron Oscillations

If we displace a layer of electrons by a distance ξ in x direction, then the change in density of electrons is given by $\delta n = n\frac{\mathrm{d}\xi}{\mathrm{dx}}$

Originally the net charge is zero, so after the displacement, Poisson's equation gives

$\frac{\mathrm{dE}}{\mathrm{dx}} = 4\pi e\delta n$

eliminating δn, we get

$\frac{\mathrm{dE}}{\mathrm{dx}} = 4\pi ne\frac{\mathrm{d}\xi}{\mathrm{dx}}$

Integrating, we get the electric field as $E = 4\pi ne\xi$

Hence, for the restoring force on the electrons, we get

$m_e\xi'' = -4\pi ne^2\xi$

This is the equation for simple harmonic motion. The frequency of oscillation is $\nu_e = \sqrt{\frac{\mathrm{ne}^2}{\pi \mathrm{m_e}}} = 8980\sqrt{n}$

Thus, displacement of a layer of electrons gives rise to a collective phenomena in plasma, that of oscillations of the displaced charges. These oscillations are in the direction of propogation vector, and hence the electric field also points in the same direction. Also, note that $\frac{\mathrm{d}\omega}{\mathrm{dk}} = 0$ hence these are not travelling waves, no energy is transported.

4.2 Plasma Ion Oscillations

Electric forces of same magnitude act of ions, but the ions being 2000 times heavy, experience that much less acceleration. In the case of Ion oscillations, the cutoff frequency turns out to be two orders of magnitude lower.

$\nu_e \simeq 9 * 10^8$, whereas $nu_p \simeq 1.5 * 10^6$

The ion oscillations essentially behave like electrostatic sound waves.

If we define the electron thermal speed to be $v_e \equiv (kT/m_e)^{1/2}$, then $\omega_p \equiv v_e/\lambda_D$. Thus, thermal electron travels about a Debye length in a plasma period. Just as the Debye length functions as the electrostatic correlation length, so the plasma period plays the role of the electrostatic correlation time.

5 Warm Plasma Waves

Note that the Plasma oscillations discussed in earlier section are not travelling waves, but stationary oscillations at a particular frequency, ω_p. These oscillations become propagating waves when we take the electron pressure to be non-zero. For the electron pressure, we can use the adiabetic relation $p_e = Cn_e^\gamma$.

The Euler equation is

$$m_e n_e \left[\frac{\partial \vec{V_e}}{\partial t} + (\vec{V_e} \cdot \nabla)\vec{V_e}\right] = -\nabla p_e + q_e n_e \left(\vec{E} + \frac{\vec{V_e}}{c} \times \vec{B}\right) \tag{15}$$

Linearising, we get

$$m_e n_0 \frac{\partial \vec{V_e}}{\partial t} = -\frac{\gamma p_0}{n_0}\nabla n_1 + e n_0 \vec{E_1} \tag{16}$$

The equation of continuity is

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \vec{V_e}) = 0 \tag{17}$$

Linearizing, we get

$$\frac{\partial n_1}{\partial t} + n_0 \nabla \cdot \vec{V_1} = 0 \tag{18}$$

From Poisson's equation,

$$\nabla \cdot \vec{E}_1 = -4\pi e n_1 \tag{19}$$

Above three linearized equations involve the three perturbation variables $n_1, \vec{V}_1$ and $\vec{E}_1$. To proceed further, we assume all these variables to vary in space and time as $exp[i(kx - \omega t)]$. The x component of the Euler eqn becoems,

$$-i\omega m_e n_0 v_{1x} = -e n_0 E_{1x} - i\frac{\gamma p_0}{n_0} k n_1 \tag{20}$$

and the other two equations give

$$-i\omega n_1 + i n_0 k v_{1x} = 0 \tag{21}$$

$$ikE_{1x} = -4\pi e n_1 \tag{22}$$

On combining these, we obtain the dispersion relation as

$$\omega^2 = \omega_p^2 + k^2 \frac{\gamma p_0}{m_e n_0} \tag{23}$$

Taking γas 3 for longitudinal one dimentional electrostatic waves, and writing $p_0 = \kappa_B n_0 T$, we can rewrite the dispersion relation as

$$\omega^2 = \omega_p^2 + k^2 \frac{3\kappa_B T}{m_e} \tag{24}$$

It is easy to see that the group velocity is

$$v_{gr} = \frac{\mathrm{d}\omega}{\mathrm{dk}} = \frac{3\kappa_B T}{m_e} \frac{k}{\omega} \tag{25}$$

The waves are therefore propagating as long as the temperature is non-zero.

6 Measurements of distances of Pulsars

The dispersion of frequencies after passing through a region of plasma can give rise to important observations in astrophysical phenomena.

A pulsar is a rotating neutron star, which emits pulses of radio waves at periodic intervals.

Waves with different frequencies travel with different group velocities in a plasma. As we can see,

$$v_{gr} = \frac{\mathrm{d}\omega}{\mathrm{dk}} = c\sqrt{1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2}} \tag{26}$$

The pulse therefore has a spread in the arrival times of waves with different frequencies. The arrival time is

$$t_p = \int_0^L \frac{dl}{v_{gr}} = \frac{1}{c}\int_0^L (1 + \frac{\omega_p^2}{\omega^2}) dl \tag{27}$$

Where we have substituted for v_{gr} and made a binomial expansion in the small quantity ω_p^2/ω^2. On substituing for ω_p^2, we get

$$t_p = \frac{L}{c} + \frac{2\pi e^2}{m_e c\omega^2}\int_0^L n_e dl \tag{28}$$

The spread in arrival times of the waves with different frequencies is then given by

$$\frac{dt_p}{d\omega} = -\frac{4\pi e^2}{m_e c\omega^3}\int_0^L n_e dl \tag{29}$$

From the spread in arrival times, on can then calculate the quantity

$$\int_0^L n_e dl = \langle n_e \rangle L \tag{30}$$

which is known as *Dispersion Measure*. Since the electron density in the interstellar medium in the solar neighbourhood is about $\langle n_e \rangle \approx 0.03cm^{-3}$, one can estimate the distance of the pulsar from the dispersion measure.

These plasma effects become noticeable for long-wavelength.

7 Vlasov Theory of Plasma Waves and Landau Damping

If we observe plasma over the timescales τ larger than $1/\omega_p$, then the single or two fluid models can be appropriate. But if the timescales over which the densities and pressures change is smaller than the relaxation time $\sim 1\omega_p$, then the fluid description is not valid, we have to take into account the particle description of plasma. The velocity distribution function is no more Maxwellian and temperature is not thermodynamically related to pressure or density.

In such a case we introduce a distribution function f such that $f(\vec{x}, \vec{V}, t)d^3xd^V = \#$ of particles in phase space volume d^3xd^3V.

Then, number density $n(\vec{x}, t) = \int f d^3V$, for neutral gas. If we ignore collisions, the average potential $\phi(\vec{x}, t) = 0$, considering ϕ as short range potential, which dies off quickly after an interaction distance r_{int}. If we ignore the collision terms in Boltzmann's equation, the rate of change of distribution function is given by

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \vec{V} \cdot \frac{\partial f}{\partial \vec{x}} - \frac{\partial \phi}{\partial \vec{x}} \cdot \frac{\partial f}{\partial \vec{V}} = 0 \tag{31}$$

Above equation is also called Vlasov eqn. The distribution function is a function of seven variables. The phase space is of six dimentions. along with Vlasov equation evaluated on a single electron and a single ion trajectory, we need maxwells equation and following two consistency equations, to form a complete set of equations describing plasma:

$$\rho(\vec{x}, t) = e\int (Zf_i - f_e)d^3V \tag{32}$$

$$\vec{J} = e\vec{(}Zf_i - f_e)\vec{V}d^3V \tag{33}$$

The generality of Vlasov eqn is that, for a given field distribution, if we find the constants of motion for electron and ions, say $I_i(\vec{x}, \vec{V}, t)$ and write the distribution function as function of these, say $f(I_1, I_2, \ldots$ then the Vlasov equation is solved. It then remains to self consistently find ρ and $\vec{J}$ and the fields there from.

One of the intesting consequences of the Vlasov equation formulation was found by Landau in 1964. The equation predicts that even though plasma is a collisionless system in this regime, energy can be transferred from a wave to the particles, causing damping. This is a surprising result, because we generally expect that damping should be due to collisions in which energy from the particles is converted into some other form. It turns out, that even in the collisionless waves, there are some particles which travel at the phase velocity of the wave, thus being able to absorb lot of energy from the wave in a resonant manner. This transfer is shown to be reversible, giving rise to 'Plasma Echos'.

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