Essay

SHOCK WAVES

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Contents

Abstract

In this essay I have discussed a very important aspect of the supersonic motion i.e. shock waves.Beginning from steepening into a shock wave , I have discussed the shock adiabatic and the case of a polytropic gas and then discussed a few astrophysical examples.

1 Introduction

When the velocity of a fluid in motion becomes comparable with or exceeds that of sound , effects due to compressibility of the fluid becomes of prime importance.Such motions are in practice met with in gases.The dynamics of high - speed flow is therefore usually called gas dynamics.

The flow of gas is entirely different in nature according as it is *subsonic* or *supersonic* ,i.e. the velocity is less than or greater than that of sound .One of the most distinctive features of supersonic flow is the fact that there can occur in it what are called *shock waves* , which is the topic of this essay.

2 The Propagation of Disturbances in a moving gas

If a gas in steady motion receives a slight perturbation at any point , the effect of the perturbation is subsequently propagated through the gas with the velocity of sound (relative to the gas itself).The rate of the propagation of the disturbance relative to a fixed system of coordinates is composed of two parts : firstly, the perturbation is "carried along" by the gas flow with velocity $\mathbf{v}$ and , secondly , it is propagated relative to gas with the velocity $\mathbf{c}$ in any direction $\mathbf{n}$. Let us consider a uniform flow of gas with velocity $\mathbf{v}$,subjected to a small perturbation at some point O (fixed in space).The velocity $\mathbf{v}+c\mathbf{n}$ with which the perturbation is propagated from O (relative to the fixed system of coordinates) has different values for different directions of the unit vector £n.We obtain all its possible values by placing one end of the vector $\mathbf{v}$ at the point O and drawing a sphere with radius c centered at the other end.The vectors from O to points on this sphere give the possible magnitudes and directions of the velocity of propagation of the perturbation.Let us first suppose that $v < c$.Then the vector $\mathbf{v}$ + c $\mathbf{n}$ can have any direction in space(Fig. 1).That is , a disturbance which starts from any point in a subsonic flow will eventually reach every point in the gas.If , on the other hand,$v > c$,the direction of the vector $\mathbf{v}$ + $c\mathbf{n}$ can lie only in a cone with its vertex at O, which touches the sphere with its centre at the other end of the vector $\mathbf{v}$. If the aperture angle of the cone is 2 α, then, as is seen from the figure (1),

$$sin\alpha = c/v \tag{1}$$

Thus a disturbance starting from any point in a supersonic flow is propagated only downstream within a cone whose aperture angle decreases with the ratio c/v.A disturbance starting from O does not affect the flow outside this cone.

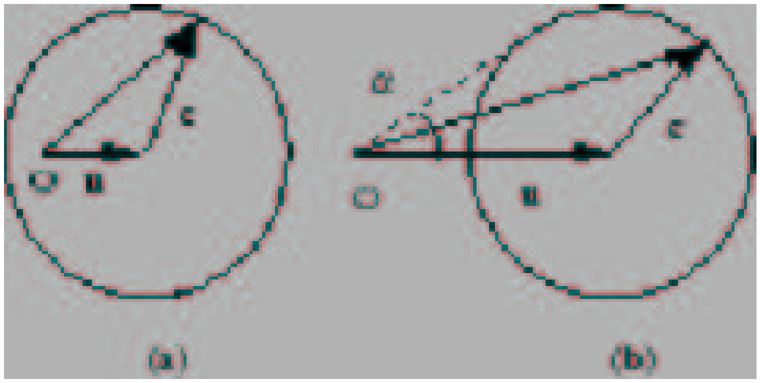

Figure 1:

The angle α determined by equation (1) is called the *Mach angle*.The ratio v/c is the *Mach number* :

$$\mathbf{M} = v/c \tag{2}$$

The surface bounding the region reached by a disturbance starting from a given point is called the *Mach surface* or *characteristic surface.*

3 Steepening into Shock Waves

Let us consider a piston being pushed into a tube of gas (Fig (2)).We assume that the piston is being moved into the gas at supersonic velocity.So the piston moves so rapidly that the gas behaves adiabatically.As the piston starts from rest with a small displacement ,the gas next to the piston gets slightly compressed .As the compression is a perturbation ,this perturbation travels as a signal into the gas at the sound speed .As the piston continues to move into the gas ,the gas next to piston is further compressed and a further signal travels into the already compressed gas ; this signal travels at the sound speed of the already compressed gas.Because the sound speed $c_s^{ad} \propto \rho^{1/3}$ this new sound speed is slightly higher than the original.Hence, arbitrarily diviing the gas in front of the piston into zones labelled 1,2,3,..., we see that the densities $\rho(1)$,$\rho(2)$,$\rho(3)$,... in these zones obey $\rho(1) > \rho(2) > \rho(3) >$ ..., so that the sound speed $c_s^{ad}(1) > c_s^{ad}(2) > c_s^{ad}(3)$...

.Thus , ’news’ of the piston travels into the gas at c_s^{ad} , and gas sufficiently far to the right remains at its original density until this ’news’ has had time to arrive. Hence, the denser zones to the left continually try to catch up the lower density zones to the right; the compression of the left hand zones continually increases, and the density profile in front of the piston therefore steepens (Fig. 3)

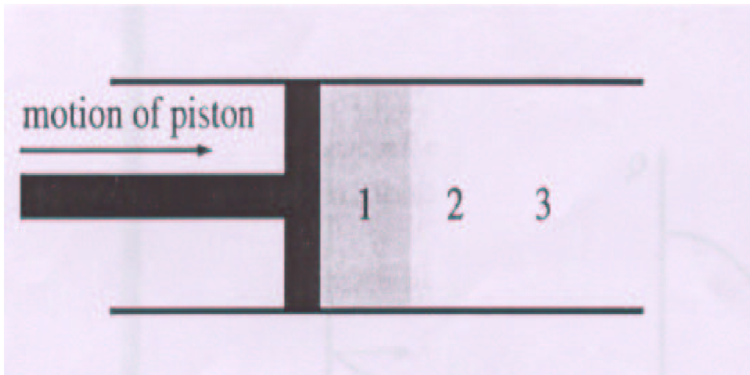

Figure 2: Piston moving at supersonic speed into the gas

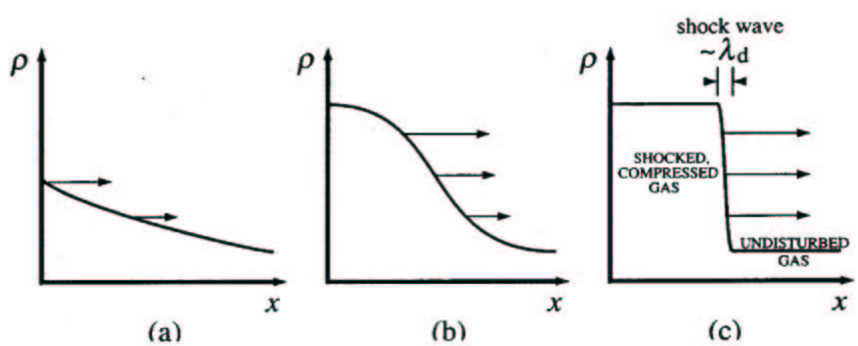

Figure 3: Steepening of the density profile leading to formation of the shock waves

But the steepening of the density profile cannot continue indefinitely. When the gradients exist in the flow over a distance scale of the order of the mean-free-path,microscopic processes - particularly those giving rise to viscous forces must be included in the momentum equation .(Thermal conduction also becomes important in the energy equation under these circumstances.)The viscous stress converts the fluid's ordered , bulk kinetic energy into microscopic kinetic energy, i.e. thermal energy.The ordered fluid velocity **v** thereby is rapidly -almost discontinuously-reduced from supersonic to subsonic, and the fluid is heated. The cooler, supersonic region of incoming fluid is said to be *ahead of* or *upstream from* the shock; the hotter , subsonic region of outgoing fluid is said to be *upstream from* the shock.The discontinuous profile given in fig(3) can be thought of as the eventual result. This surface at which the pressure (and also density and velocity) change discontinuously is called a shock wave.In many cases; however , a detailed picture of the shock is not required; since the shock thickness (λ_d is much smaller than the lengthscales of gradients in the gas on each side of it, we can approximate the shock as a discontinuity in the gas flow. The connection between the gas density, velocity and pressure (or temprature) across this idealised discontinuity can be found by applying conservation laws.

4 Surfaces of Discontinuity

The rate of motion of these surfaces of discontinuity bear no relation to the velocity of the gas itself.The gas particles in their motion may cross a surface of discontinuity.

Certain boundary conditions must be satisfied on the surfaces of discontinuity To formulate these conditions , we consider an element of the surface and use a coordinate system fixed to this element,with the x-axis along the normal.

Firstly, the mass flux must be continuous : the mass of gas coming from one side must equal the mass leaving the other side.The mass flux through the surface element considered is ρv_x per unit area.Hence we must have $\rho_1 v_{1x} = \rho_2 v_{2x}$,where the suffixes 1 and 2 refer to the two sides of the surface of discontinuity.

The difference between the two values of any quantity on the two sides of the surface will be denoted by enclosing it in square brackets; for example, $[\rho v_x] = \rho_1 v_{1x} - \rho_2 v_{2x}$,and the conditionjust derived can be written

$$[\rho v_x] = 0 \tag{3}$$

Next, the energy flux must be continuous .We therefore obtain the condition

$$[\rho v_x(\frac{1}{2}v^2 + w)] = 0 \tag{4}$$

Here w is the heat function per unit mass of the fluid.

Finally, the momentum flux must be continuous , i.e. the forces exerted on each other by the gases on the two sides of the surface of discontinuity must be equal. The momentum flux per unit area is $pn_i + \rho v_i v_k n_k$. The normal vector **n** is along the x-axis. The continuity of the x-component of the momentum flux therefore gives the condition

$$[p + \rho v_x^2] = 0, \tag{5}$$

while that of the y and z components gives

$$[\rho v_x v_y] = 0, [\rho v_x v_y] = 0 \tag{6}$$

Equations (3)-(6) form a complete system of boundary conditions at a surface of discontinuity.From them we deduce the possibility of two types of discontinuity.

In the first type , there is no mass flux through the surface . This means that$\rho_1 v_{1x} = \rho_2 v_{2x} = 0$. Since ρ_1 and ρ_2 are not not zero,it follows that $v_{1x} = v_{2x} = 0$.The conditions (4) and (6) are then satisfied, and the condition (5) gives $p_1 = p_2$.Thus the normal velocity component and the gas pressure are continuous at the surface of discontinuity :

$$[v_{1x} = v_{2x}] = 0, [p] = 0, \tag{7}$$

while the tangential velocities v_y, v_z and the density may be discontinuous by any amount.This is called a *tangential discontinuity*.

In the second type, the mass flux is not zero, and v_{1x} and v_{2x} are therefore also not zero.From (3) and (6)

$$[v_y] = 0, [v_z] = 0, \tag{8}$$

i.e, the tangential velocity is continuous at the surface of discontinuity.The pressure , the density and the normal velocity , however,are discontinuous, their discontinuities being related by (3)-(5).In the condition (4) we can cancel ρv_x by (3),and replace v^2 by v_x^2 since v_y and v_z are continuous. Thus the following conditions must hold at the surface discontinuity in this case:

$$[\rho v_x] = 0 \tag{9}$$

$$[\frac{1}{2}v^2 + w] = 0 \tag{10}$$

$$[p+\rho v_x^2]=0, \tag{11}$$

A discontinuity of this kind is called a *shock wave*,or simply a *shock* .

If we now return to the fixed coordinate system , we must everywhere replace v_x by the difference between the gas velocity component v_n normal to the surface of surface:

$$v_x = v_n - u \tag{12}$$

The velocities v_n and u are taken in the fixed system.The velocity v_x is the velocity of the gas relative to the surface of discontinuity ; we can also say that $-v_x = u - v_n$ is the rate of propagation of the surface relative to the gas.If v_x is discontinuous,this velocity has different values relative to the gas on the two sides of the surface.

5 The Shock Adiabatic

In this type of discontinuity,the tangential component of the gas velocity is continuous. We can therefore take a coordinate system in which the surface element considered is at rest, and the tangential component of the gas velocity is zero on both sides.Then we can write the normal component v_x as v simply, and conditions (9),(10),(11) take the form

$$\rho_1 v_1 = \rho_2 v_2 = j \tag{13}$$

$$\frac{1}{2}\rho_1 v_1^2 + w_1 = \frac{1}{2}\rho_2 v_2^2 + w_2 \tag{14}$$

$$p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2 \tag{15}$$

where j denotes the mass flux density at the surface of discontinuity . We take j positive , with the gas going from side 1 to side 2.That is, gas 1 is the one into which the shock wave moves, and gas 2 remains behind the shock.The side of the shock wave towards gas 1 is the *front* of the shock, and that towards gas 2 is the *back*.

Using the specific volumes $V_1 = 1/\rho_1, V_2 = 1/\rho_2$,we obtain from (13)

$$v_1 = jV_1, \;\; v_2 = jV_2, \tag{16}$$

and , substituting in (14),

$$p_1 + j^2 V_1 = p_2 + j^2 V_2, \tag{17}$$

or

$$j^2 = (p_2 - p_1)/(V_1 - V_2), \tag{18}$$

This formula, together with (16), relates the rate of propagation of a shock wave to the pressures and densities of the gas on the two sides of the surface.

Since j^2 is positive, we see that either $p_2 > p_1$,$V_1 > V_2$,or $p_2 < p_1$,$V_1 < V_2$;we shall see that only the former case can actually occur.

We note the following formula for the velocity difference $v_1 - v_2$

$$v_1 - v_2 = j(V_1 - V_2) \tag{19}$$

Substituting (18) in (19) we obtain

$$v_1 - v_2 = [(p_2 - p_1)((V_1 - V_2)]^{1/2} \tag{20}$$

Next,we write(15) in the form

$$w_1 + \frac{1}{2}j^2V_1^2 = w_2 + \frac{1}{2}j^2V_2^2 \tag{21}$$

and,substituting j^2 from (18),obtain

$$w_1 - w_2 + \frac{1}{2}(V_1 + V_2)(p_2 + p_1) = 0. \tag{22}$$

If we replace the heat function w by $\varepsilon + pV$,where ε is the internal energy, we can write this relation as

$$\varepsilon_1 - \varepsilon_2 + \frac{1}{2}(V_1 - V_2)(p_1 + p_2) = 0. \tag{23}$$

These relations hold between the thermodynamic quantities on the two sides of the surface of discontinuity.

For given p_1, V_1,equation (22) or (23) gives the relation between p_2 and V_2.This relation is called the *shockadiabatic* or the *Hugoniotadiabatic* (W.J.M.Rankine 1870; H.Hugoniot 1885).It is represented graphically in the pV -plane(Fig 4) by a curve passing through the given point (p_1, V_1)corresponding to the state of gas 1 in front of the shock wave , which we shall call the *intialpoint*.The shock adiabatic cannot intersect the vertical line $V = V_1$ except at the intial point.For the existence of another intersection would mean that two different pressures satisfying (23) correspond to the same volume.For $V_1 = V_2$,however, we have from (23) also $\varepsilon_1 = \varepsilon_2$, and when the volumes and energies are the same the same pressures

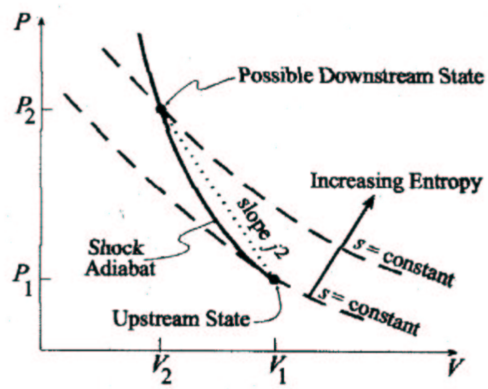

Figure 4: The shock adiabatic.

must be the same. Thus the line $V = V_1$ divides the shock adiabatic into two parts, each of which lies entirely on the side of the line.Similarly, the shock adiabatic meets the horizontal line $p = p_1$ only at the point (p_1, V_1).

Let $aa^{'}$ (Fig 5) be the shock adiabatic through the point (p_1, V_1) as intial point.We take any point (p_2, V_2) on it and draw through that point another adiabatic $bb^{'}$, for which (p_2, V_2) is an intial point.The pair of values(p_1, V_1) satisfies the equation of this adiabatic also.The adiabatics $aa^{'}$ and $bb^{'}$ therefore intersect at the two points (p_1, V_1) and (p_2, V_2). The adiabatics are not identicalThe equation of the shock adiabatic can not be written in the form $f(p, V)$ = constant, where f is some function.The shock adiabatic is determined by two parameters,the intial values p_1 and V_1.Hence, if two (or more) successive shock waves take a gas from state *1* to state *2* and from there to state *3*,the transition from state*1* to state *3* cannot in general be affected by the passage of any one shock wave.

For a given intial thermodynamic state of the gas (i.e. for given p_1 and V_1),the shock wave is defined by only one parameter;for instance, if the pressure p_2 behind the shock is given , then V_2 is determined by the Hugoniot adiabatic, and the flux density j and the velocities v_1 and v_2 are then given by formulae (16) and (18).

The formula (18) has the following convenient interpretation.If the point (p_1, V_1) on the shock adiabatic (Fig 4) is joined by a chord to any other point (p_2, V_2) on it , then $(p_2 - p_1)/(V_2 - V_1) = -j^2$ is the slope of this chord relative to the axis of abscissae. Thus j, and therefore the velocity of the shock wave are determined

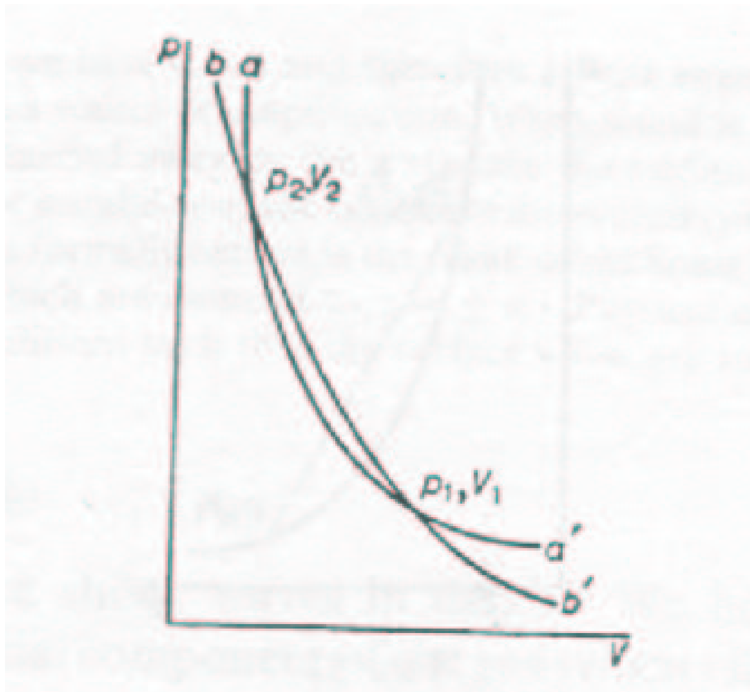

Figure 5:

at each point of the shock adiabatic by the slope of the chord joining that point to the intial point.

The entropy is discontinuous at a shock wave .By the law of increase of entropy, the entropy of a gas can only increase during its motion.Hence the entropy s_2 of the gas which has passed through the shock waves must exceed the intial entropy s_1 :

$$s_2 > s_1 \tag{24}$$

The presence of shock waves results in an increase in those flows which can be regarded as motions of an ideal fluid in all space, the viscosity and thermal conductivity being zero.The increase in entropy signifies that the motion is irreversible,i.e. energy is dissipated. Thus the discontinuities are a means by which energy can be dissipated in the motion of an ideal fluid.

The true mechanism by which the entropy increases in shock waves lies in the dissipative processes occurring in the very thin layers which actual shock waves are.The increase in entropy in a shock wave has an important effect on the motion: even if there is potential flow in front of the shock wave , the flow behind it is in general rotational.

6 Shock Waves in a Polytropic gas

We consider a shock wave in a polytropic gas.The heat function of such a gas is given by The heat function of such a a shock wave in a polytropic gas.

$$w = c_p T = pV/(\gamma-1) = c^2/(\gamma-1) \tag{25}$$

Substituting this in (22), we have Using this formula, we can determine any of the quantities p_1, V_1, p_2, V_2 from the other three.The ratio V_2/V_1 is a monotonically decreasing function of the ratio p_2/p_1, tending to the finite limit of

$$\frac{V_2}{V_1} = \frac{\rho_1}{\rho_2} = (\gamma-1)(\gamma+1) \tag{26}$$

The curve showing p_2 as a function of V_2 for given p_1, V_1 (the shock adiabatic) is represented in Fig(6).It is a rectangular hyperbola with asymptotes $V_2/V_1 = (\gamma-1)/(\gamma+1)$, $p_2/p_1 = -(\gamma-1)/(\gamma+1)$. Only the upper part of the curve , above the point $V_2/V_1 = p_2/p_1 = 1$ has physical significance.It is shown in fig (6) by a continuous line.

For the ratio of the temperatures on the two sides of the discontinuity we find, from the equation of state for a perfect gas, $T_2/T_1 = p_2V_2/p_1V_1$, that

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{(\gamma+1)p_1 + (\gamma-1)p_2}{(\gamma-1)p_1 + (\gamma+1)p_2} \tag{27}$$

For the mass flux we obtain from Eq. (18) and (26)

$$j^2 = [(\gamma-1)p_1 + (\gamma+1)p_2]/2V_1 \tag{28}$$

and then for the velocities of propagation of the shock wave ralative to the gas before and behind it

$$\begin{aligned}
v_1^2 &= \frac{1}{2}V_1[(\gamma-1)p_1 + (\gamma+1)p_2] \\
&= \frac{1}{2}\frac{c_1^2}{\gamma}\left[\gamma - 1 + (\gamma+1)\frac{p_2}{p_1}\right] \\
v_2^2 &= \frac{1}{2}V_1\frac{[(\gamma-1)p_1 + (\gamma-1)p_2]^2}{(\gamma-1)p_1 + (\gamma+1)p_2} \\
&= \frac{1}{2}\frac{c_2^2}{\gamma}\left[\gamma - 1 + (\gamma+1)\frac{p_1}{p_2}\right]
\end{aligned} \tag{29}$$

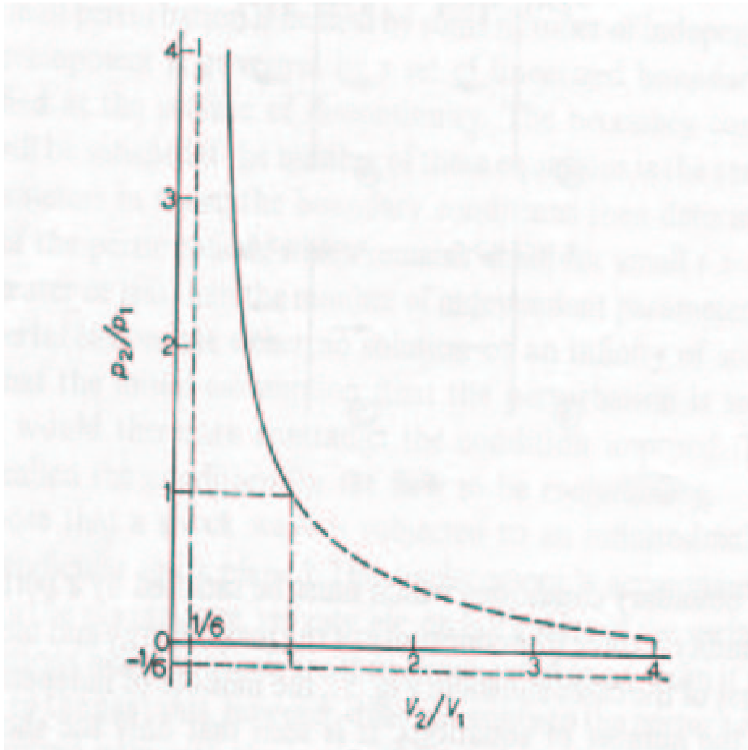

Figure 6:

their difference being

$$v_1 - v_2 = (2V_1)(p_2 - p_1)^1/2/(\gamma - 1)p_1 + (\gamma + 1)p_2. \tag{30}$$

There are some formulae useful in applications, which express the ratios of densities, pressures and tempratures in a shock wave in terms of the Mach number $M_1 = v_1/c_1$. These formulae are easily derived from the foregoing results:

$$\rho_2/\rho_1 = v_1/v_2 = (\gamma + 1)M_1^2/[(\gamma - 1)M_1^2 + 2], \tag{31}$$

$$p_2/p_1 = 2\gamma M_1^2/(\gamma + 1) - (\gamma - 1)/(\gamma + 1), \tag{32}$$

$$T_2/T_1 = [2\gamma M_1^2 - (\gamma - 1)][(\gamma - 1)M_1^2 + 2]/(\gamma + 1)^2 M_1. \tag{33}$$

The Mach number M_2 is given in terms of M_1 by

$$\mathcal{M}_2^2 = \frac{2+(\gamma-1)\mathcal{M}_1^2}{2\gamma\mathcal{M}_1^2-(\gamma-1)}. \tag{34}$$

. This is symmetrical in M_2 and M_1; it may be written in the form:

$$2\gamma\mathcal{M}_1^2\mathcal{M}_2^2-(\gamma-1)(\mathcal{M}_1^2+\mathcal{M}_2^2) = 2. \tag{35}$$

We can give the limiting results for very strong shock waves, in which $(\gamma-1)p_2$ is very large compared with $(\gamma+1)p_1$. From (27) and (28) we have

$$\frac{V_2}{V_1} = \frac{\rho_1}{\rho_2} = \frac{\gamma-1}{\gamma+1} \tag{36}$$

$$\frac{T_2}{T_1} = \frac{(\gamma-1)p_2}{(\gamma+1)p_1} \tag{37}$$

The ratio T_2/T_1 increases to infinity with p_2/p_1, i.e. the temprature discontinuity in a shock wave, like the pressure discontinuity, can be arbitrarily great. The density ratio, however, tends to a constant limit; e.g., for a momoatomic gas the limit is $\rho_2+4\rho_1$, and for a diatomic gas $\rho_2+6\rho_1$. The velocities of propagation of a strong shock wave are

$$u_1 = \left[\frac{1}{2}(\gamma+1)p_2V_1\right]^{1/2}, \quad u_2 = \left[\frac{\frac{1}{2}(\gamma-1)^2p_2V_1}{\gamma+1}\right]^{1/2}. \tag{38}$$

They increase as the square root of the pressure p_2.

Lastly, there are relations for weak shock waves, which are the leading terms in expansions in powers of the small quantity $z=(p_2-p_1)/p_1$:

$$M_1-1 = 1-M_2 = (\gamma+1)z/4\gamma, \tag{39}$$

$$c_2/c_1 = 1+(\gamma-1)z/2\gamma. \tag{40}$$

$$\rho_2/\rho_1 = 1+z/\gamma-(\gamma-1)z^2/2\gamma. \tag{41}$$

These are the terms giving the first correction to the acoustic approximation.

7 Similarity Solutions - Sedov-Taylor Blast Wave

Strong explosions can generate shock waves. Examples include atmospheric nuclear explosions, supernova explosions, and depth charges. The debris of a strong explosion will be at much higher pressure than the surrounding gas and will therefore drive a strong spherical shock into the surroundings. Intially, this shock wave will travel at roughly the radial speed of the expanding debris. However the mass of fluid swept up by the shock will eventually exceed that of the explosion debris. The shock will decelerate and the energy of the explsion will be transferred to the swept-up fluid.Let us calculate how fast and how far the shock front will travel.

First let us make an order of magnitude estimate. Let the total energy of the explosion be E and the density of the surrounding fluid (assumed uniform) be ρ_0. Then after time t, when the shock radius is $R(t)$, the mass of the swept-up fluid will be $\approx \rho_0 R^3$. The fluid velocity behind the shock will be roughly the radial velocity of the shock front, $v \approx \dot{R} \approx R/t$, and so the kinetic energy of the swept-up mass will be $\approx \rho_0 R^5/t^2$. There will also be the internal energy in the post-shock flow, with energy density roughly equal to the post-shock pressure, $\rho\varepsilon \approx P \approx \rho_0 \dot{R}^2$ [cf. the strong-shock jump conditions with $P_1 \approx \rho_0 c_0^2$ so $P_1 M^2 \approx \rho_0 v^2 \approx \rho_0 \dot{R}^2$]. The total internal energy within the expanding shock will then be $\approx \rho R^2 \dot{R}^2$, equal in order of magnitude to the kinetic energy. Equating either term to the total energy E of the explosion, we obtain the rough estimate

$$E = K\rho_0 R^5 t^{-2}, \tag{42}$$

which implies that at time t the shock front has reached the radius

$$R = (\frac{E}{K\rho_0})^{1/5} t^{2/5}. \tag{43}$$

Here K is a numerical constant of order unity. This scaling should hold roughly from the time that the mass of the debris is swept up to time that the shock weakens to a Mach number of order unity so we can no longer use the strong-shock value $\approx \rho_0 \dot{R}^2$ for the post shock pressure.

If we assume the motion remains radial and the gas is perfect with constant specific-ratio γ, then we can solve for the details of the flow behind the shock front by integrating the radial flow equations

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho v) = 0, \frac{\partial v}{\partial t} + \frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial P}{\partial r} = 0, \frac{\partial}{\partial t}(\frac{P}{\rho^\gamma}) + v\frac{\partial}{\partial r}(\frac{P}{\rho^\gamma}) = 0. \tag{44}$$

The first two equations are the continuity equation and Euler equation written for a spherical flow. The third equation is energy conservation expressed as the adiabatic-expansion relation, $P\alpha\rho^{\gamma}$ moving with a fluid element. although $\frac{P}{\rho^{\gamma}}$ is time-independent for each fluid element, its value will change from element to element. Gas that has passed through shock more recently will be given a smaller entropy than gas which was swept up when the shock was stronger, and thus have a smaller value of $\frac{P}{\rho^{\gamma}}$.

Given suitable intial conditions, the above partial differential equations can be integrated numerically. However, there is a practical problem in that it is not easy to determine the intial conditions in an explosion! Fortunately, at late times, when most of the mass has been swepr up, the fluid evolution is independent of the details of the intial expansion and can be understood analytically as a *similarity solution*. By this, it means that the shape of the radial profiles of pressure, density and velocity are independent of time.

The characteristic scaling length in the explosion is the radius $R(t)$ of the shock, so the fluid and thermodynamic variables should be expressible as some characteristic values multiplying universal functions of

$$\xi = r/R(t). \tag{45}$$

Our thermodynamic variables are P, ρ, ε and a natural choice for their characteristics values is the values immediately behind the shock. If we assume that the shock is strong then we can use the strong-shock jump conditions to determine those values and write

$$P = \frac{2}{\gamma+1}\rho_0\dot{R}^2 f(\xi), \rho = \frac{\gamma+1}{\gamma-1}\rho_0 g(\xi), \tag{46}$$

$$v = \frac{2}{\gamma+1}\dot{R}h(\xi) \tag{47}$$

with f(1) = g(1) = h(1) since $\xi = 1$ is the shock's location. The velocity v is scaled to the post-shock velocity measured in the inertial frame in which the upstream fluid is at rest, rather than in the non-inertial frame in which the declaration shock is at rest. The self-similarity anszatz(47)and the resulting self similar solution for the flow are called the *Sedov-Taylor blast-wave solution,* since L.I.Sedov and G.I Taylor independently developed it.

We need one more piece of information before we can solve for the flow: the variation of shock radius R with time. However all that is necessary is the scaling

$R = (E/K\rho_0)^{1/5} t^{1/5} \propto t^{2/5}$[eq 43] with the constant K left undetermined for the moment. The partial differential equations(44) can then be transformed into ordinary differential equations by inserting the anszatz(47), changing the independent variables from r,t to R,ξ and using

$$\left(\frac{\partial}{\partial t}\right)_r = -\left(\frac{\xi \dot{R}}{R}\right)\left(\frac{\partial}{\partial \xi}\right)_R + \dot{R}\left(\frac{\partial}{\partial R}\right)_\xi = -\left(\frac{2\xi}{5t}\right)\left(\frac{\partial}{\partial \xi}\right)_R + \frac{2R}{5t}\left(\frac{\partial}{\partial R}\right)_\xi \tag{48}$$

$$\left(\frac{\partial}{\partial r}\right)_t = \left(\frac{1}{R}\right)\left(\frac{\partial}{\partial \xi}\right)_R \tag{49}$$

The three resulting first order deifferential equations are rather complex but can infact be solved analytically (e.g Landau and Lifshitz 1959). The results for an explosion in air are exhibited in fig (7).

Using these solutions for $f(\xi), g(\xi), h(\xi)$, we can evaluate the flows's energy during time interval when this similarity solution is accurate. The energy E is given by the integral

$$E = \int_0^R 4\pi r^2 dr \rho \left(\frac{1}{2}v^2 + \varepsilon\right) \tag{50}$$

$$= \frac{4\pi\rho_0 R^3 \dot{R}^2(\gamma+1)}{\gamma-1}\int_0^1 d\xi \xi^2 g\left(\frac{2h^2}{(\gamma+1)^2} + \frac{2h^2}{(\gamma+1)^2 g}\right) \tag{51}$$

Here I have used eqn (47) and substituted $\varepsilon = P/\rho(\gamma-1)$for the internal energy. The energy E appears not only on the left side of this equation, but also on the right, in the terms $\rho_0 R_3 \dot{R}^2 = (4/25)E/K$. Thus, E cancel out, and the equation (51) becomes an equation for the unknown constant K. Evaluating that equation numerically, we find that K varies from 2.5 to 1.4 as γ increases from 1.4(air) to 1.67(monatomic gas or fully ionised plasma).

Let us see how the fluid behaves in this blast-wave solution. The fluid that passes through the shock is compressed so that it almost occupies a fairly thin spherical shell immediately behind the shock. This shell moves somewhat slower than the shock[$v = 2\dot{R}/(\gamma+1)$; Eq.(47) and fig (7). As the post shock flow is subsonic, the pressure within the blast wave is fairly uniform; infact the central pressure is typically half the maximum pressure immediately behind the shock. This pressure on the spherical shell, thereby accelerating the freshly swept-up fluid.

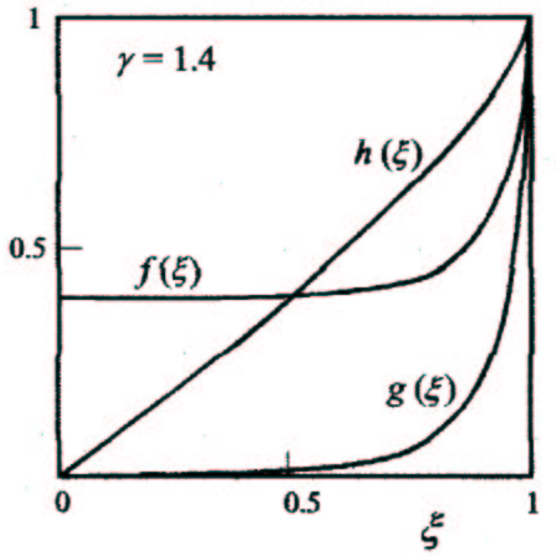

Figure 7: Scaled pressure, density and velocity as a function of scaled radius behind the Sedov-Taylor blast wave.

7.1 Supernovae

The evolution of most massive stars ends in supernova explosion in which a neutron star of mass m $\sim 3\times10^{30}$ kg is formed. This neutron star has a gravitational binding energy of about $0.1mc^2 \sim 3\times10^{46}$ J. Most of this binding energy is released in the form of neutrinos in the collapse that forms the neutron star, but an energy $E \sim 10^{44}$J drives off the outer envelope of the presupernova star, a mass $M_0 \sim 10^{31}kg$. This stellar material escapes with rms speed $V_0 \sim 2E/M_0^{1/2} \sim 5000kms^{-1}$. The expanding debris eventually drives a blast wave into the surrounding interstellar medium of density $\rho_0 \sim 10^{-21}kgm^{-3}$. The expansion of the blast wave can be modeled using the Sedov-Taylor solution after the swept-up interstellar gas has become large enough to dominate the blast wave, so the star dominated initial conditions are no longer important-i.e, after a time $\sim (3M_0/4\pi\rho_0)^{1/3}/V_0 \sim 1000yr$.The blast wave then decelerates in a Taylor-Sedov self similar way until the shock speed nears the the sound speed in the surrounding gas; this takes about 100000yr.*Supernova remnants* of this sort are efficient

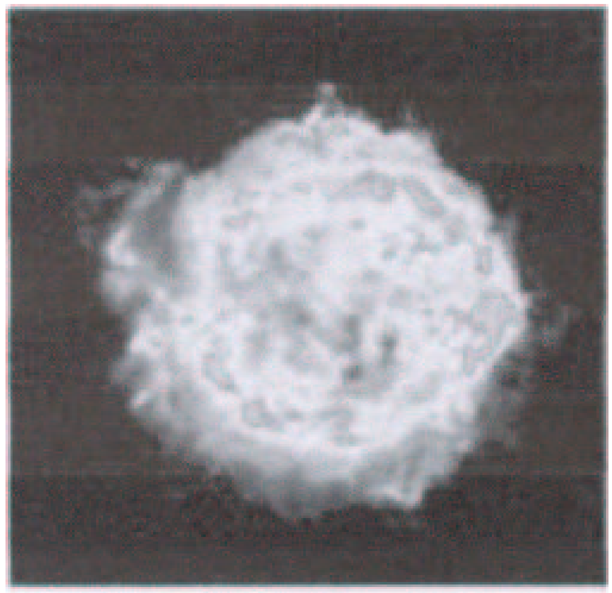

Figure 8: Cassiopeia A - a supernova remnant left behind an exploring star in our galaxy.

emitters of radio waves and several hundred have been observed in the Galaxy.

In some of the younger examples, like Cassiopiea A, (fig 8) it is possible to determine the expansion speed , and the effects of decceleration can be measured. The observations are consistent with the prediction of the Sedov-Taylor solution, namely that the radius varies as $R \propto t^{2/5}$ or $\dot{R} = -3\dot{R}^2/2R$

8 Isothermal Shocks

Now we consider the case when the shocked gas cools by radiation.We shall consider only the extreme case of very efficient cooling which allows the basic conditions of section (5) to be still satisfied .Further, we shall assume that the gas returns to its original pre-shock temprature.This situation is depicted schematically in fig (9).

Gas flows into shock S across which the flow variables are related by the standard Rankine-Hugoniot conditions. There is then a region in which cooling by

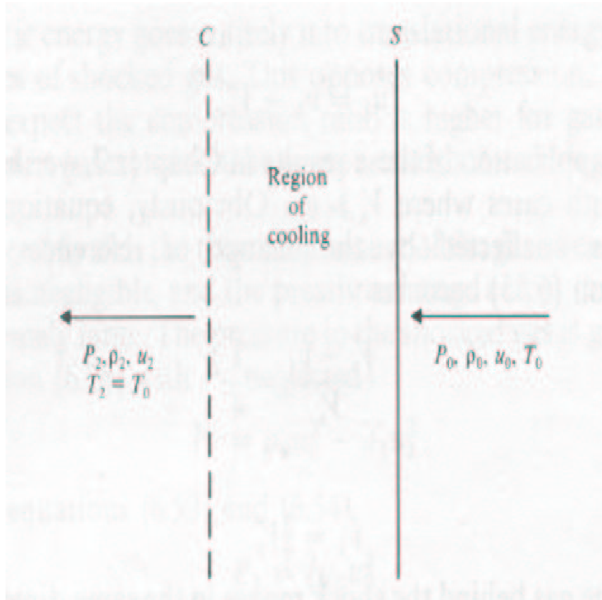

Figure 9: Schematic representation of an isothermal shock wave.

radiation occurs.C is a surface at which we assume the gas has returned to pre-shock temprature. if S and C are close enough, gas flows so quickly between S and C that the entire region can be regarded as thin and the flow time-independent Surfaces S and C can then be considered to form one surface across which the density and other parameters change but the temprature remains constant. This is called an isothermal discontinuity or an isothermal shock wave.

Let subscripts 2 refer to the downstream conditions beyond C fig (9) and subscripts 0 to the upstream conditions . Under the circumstances outlined above, the continuity and momentum conservation equations are the same as for a normal shock. The change occurs in the energy equation.Instead of summing over the various forms of energy, which would now include radiation, we simply write

$$T_0 = T_2 = constant. \tag{52}$$

The sound speed is now

$$c^2 = \frac{P}{\rho} = \frac{kT}{\mu m} \tag{53}$$

where we c is the isothermal sound speed. If c_0 is the sound speed at temprature

T_0, the condition of isothermality across the discontinuity can be written as

$$\frac{P_0}{\rho_0} = \frac{P_2}{\rho_2} = c_0^2. \tag{54}$$

We assume that the shock is strong, i.e. $\rho_0 u_0^2 \gg P_0$. The momentum equation then becomes on using (54)

$$\rho_2 c_0^2 = \rho_0 u_0^2 - \rho_2 u_2^2 \tag{55}$$

Hence from the continuity condition,

$$u_2^2 - u_2 u_0 + c_0^2 + c_0^2 = 0. \tag{56}$$

The solution of this is

$$u_2 = (\frac{u_0}{2})(1 \pm \sqrt{(}1 - \frac{4c_0^2}{u_0^2}) \tag{57}$$

Since we have assumed the shock to be strong, $u_0 \gg c_0$ and thus

$$u_2 = (\frac{u_0}{2})(1 \pm (1 - \frac{2c_0^2}{u_0^2}). \tag{58}$$

The positive sign gives the trivial solution $u_2 \approx u_0$. Since we know compression must occur across the adiabatic shock, we must take the negative sign and obtain

$$u_2 \approx \frac{c_0^2}{u_0}. \tag{59}$$

The upstream Mach number, which is now defined with respect to the isothermal sound speed is

$$M_0 = \frac{u_0}{c_0}. \tag{60}$$

Thus we can write using the continuity equation and equation (59)

$$\frac{u_0}{u_2} = \frac{\rho_2}{\rho_0} = M_0^2. \tag{61}$$

We now see tha the compression across a strong isothermal shock depends on the upstream Mach number, in contrast with the case of a strong adiabatic shock for which the compression is limited to a factor of 4. The physical reason for this is that in order that the gas remain isothermal, internal energy has to be radiated away. This internal energy would otherwise have limited the compression.

The post-shock pressure is now given by

$$P^2 = \rho_2 c_0^2 = \rho_0 u_0^2. \tag{62}$$

We express these results in terms of velocities measured in a fixed frame of reference, i.e. one in which the shock is moving.The necessary transformation is affected by the relationships

$$u_0 = v_0 - V_s \tag{63}$$

and

$$u_2 = v_2 - V_s. \tag{64}$$

Here V_s is the shock velocity in the fixed frame.v_0 and v_2 respectively be the upstream and downstream gas velocities, measured in the fixed frame. We will assume that $V_S \gg v_0$. The compression ratio ρ_2/ρ_0 remains unchanged .The velocity ratio is

$$\frac{V_S}{V_S - v_2} = M_0^2 \tag{65}$$

Thus

$$v_2 = V_S(1 - 1/M_0^2) \approx V_s \tag{66}$$

From equation (66) the post-shock pressure is

$$P_2 = \rho_0 V_s^2 \tag{67}$$

The gas behind a strong isothermal shock moves in the same direction and with the same velocity as the shock.

9 Spherical acceretion and winds

The corona of a star can be in static equlibrium only if there is some finite pressure at infinity to stop it from expanding.If the presure at infinity is less, then there will be an outward flow.On the other hand , if the pressure at infinity is more than what is needed to maintain static equlibrium ,then there would be an inward flow.Parker(1958) first perdicted a wind from the sun and worked out a spherically symmetric model for it .The first spherically symmetric inward accretion model was developed by Bondi(1952). Since spherical wind and spherical accretion are very similar problems, we present them together.When a gravitating object accretes matter from the space around it, most often the accreting matter has a non-isotropic distribution around the gravitating centre and possesses angular momentum leading to the formation of an accretion disk.Although the solar wind is found to be somewhat non-isotropic (mainly due to the presence of magnetic fields in the solar atmosphere) and the same is expected for the wind from other stars , a spherical wind model probably has more connections with reality than a spherical accretion model.

Let us consider a steady spherical flow such that velocity v is independent of time and is in the radial direction (either inward or outward). Under steady conditions, the same mass flux has to flow through spherical surfaces at different distances.Hence,

$$r^2\rho v = constant \tag{68}$$

from which

$$\frac{2}{r} + \frac{1}{\rho}\frac{d\rho}{dr} + \frac{1}{v}\frac{dv}{dr} = 0 \tag{69}$$

Assuming the gravitational field to be produced by a central mass **M** , the Euler equation gives

$$\rho v\frac{dv}{dr} = -\frac{dp}{dr} - \frac{GM\rho}{r^2} \tag{70}$$

We now have to solve (69) and (70) with an appropriate energy equation . The problem is simplest if we replace the energy equation by the assumption of an isothermal condition,i.e. if the pressure is taken to be $p = R\rho T$ with T as a constant.(69) and (70) remain invariant when we make the transformation v to -v. In other word, steady spherical wind and steady spherical accretion are mathematically the same problem. Once we have a solution for a spherical wind, we can immediately get a solution for a spherical accretion by reversing the velocity at all points.This symmetry between spherical accretion and wind holds only in the steady state. Time - dependent spherical acretion and wind problems are no longer symmetric.

par We write $p = v_c^2\rho$ in (70) where $v_c^2 = RT$ is taken as a constant.v_c is the isothermal sound speed. We eliminate ρ from (69) and (70), which gives

$$(v - \frac{v_c^2}{v})\frac{dv}{dr} = \frac{2v_c^2}{r} - \frac{GM}{r^2} \tag{71}$$

It is possible for v to be equal to v_c only at the distance

$$r = r_c = \frac{GM}{2v_c^2} \tag{72}$$

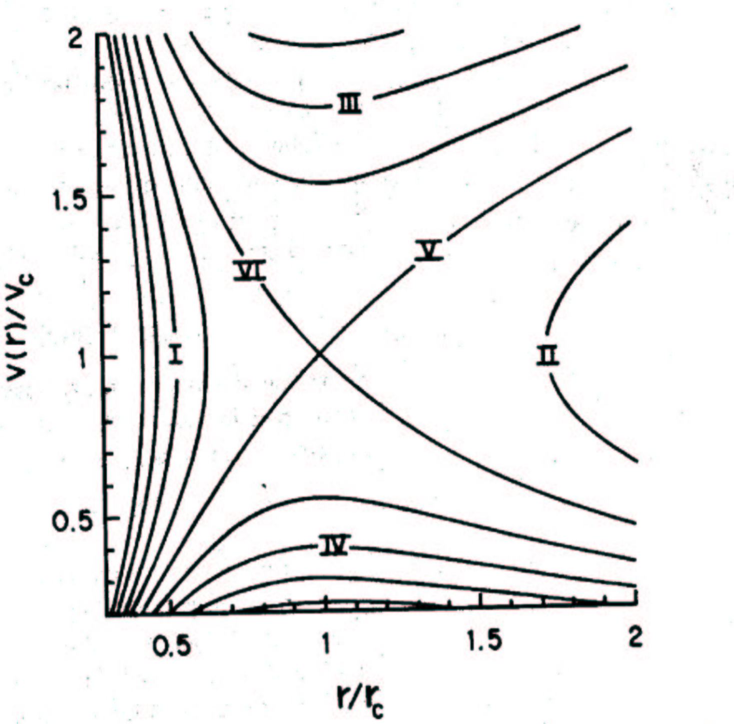

Figure 10: Different solutions $v(r)$ aaaas given by the equation(73).

so that both sides of (71) are zero simultaneously.We integate (71) to obtain

$$(\frac{v}{v_c}^2) - log(\frac{v}{v_c}^2) = 4log\frac{r}{r_c} + \frac{2GM}{rv_c^2} + C, \tag{73}$$

where C is the constant of integration. The solutions for the different values of C are fhown in fig (10). It turns out that the solutions of types I and II are double-valued and hence unphysical. The solutions of type III are supersonic everywhere, whereas solutions of type IV are subsonic everywhere(here we use the words 'subsonic' and 'supersonic' with respect to the isothermal sound speed v_c). Only solutions of type V and VI pass through the critical point $r = r_c$, $v = v_c$, and are subsonic and supersonic in different regions. We find from (73) that C = -3 for these solutions. Which particular solution is realized in a given situation again depends on the boundary conditions Parker (1958) considered the solar wind to start from subsonic speeds near the solar surface and then to accelerate to high speeds. Hence the solution V is appropriate for Parker's problem. On the other hand, Bondi(1952) studied an inflow starting from small speeds at infinity and becoming faster in the interior .So VI corresponds to the solution for Bondi's problem.

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