A Metric for the Universe

Averaged over a large enough scale, say a few hundred Mpc, the distribution of matter in the universe looks very smooth, homogeneous and isotropic. The matter-energy content in the homogeneous isotropic universe determines the gravitational field at any point in the universe. This can be described by a metric that has the features of homogeneity and isotropy, but allows for curvature. We also know that the universe may evolve in time, so the metric should allow for an explicit time dependence as well.

At any given time, let dl^2 represent the spatial part of the metric. The property of homogeneity and isotropy at all times demands that the only form of time dependence the metric may have is

$$ds^2 = c^2dt^2 - a^2(t)dl^2$$

where $a(t)$ is a scale factor.

The spatial part of the metric must be such that any point in space is equivalent. This means that the curvature of space must be constant everywhere, and this curvature could be either positive (spherical), zero (flat) or negative (hyperboloidal).

To deduce the form of the metric, let us start with a two-dimensional analogy. A two dimensional space is a surface, which could be either flat or curved. Coordinates and the metric are defined only in terms of distances measurable on the surface, and not out of it.

If the surface is flat, then we have the familiar relation

$$dl^2 = dr^2 + r^2d\phi^2$$

where r, ϕ are coordinates defined on the plane.

If the surface is curved in the form of a sphere, then the squared distance in terms of a 3-dimensional coordinate system defined with the origin at the centre of the sphere is

$$dl^2 = dR^2 + R^2d\theta^2 + R^2\sin^2\theta d\phi^2.$$

For distances between points lying on the spherical surface of radius R we have $dR = 0$, and hence

$$dl^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2.$$

We can always choose an unit of length such that $R = 1$. Then

$$dl^2 = d\theta^2 + \sin^2\theta d\phi^2$$

Let us now introduce coordinates r, ϕ on the surface of the sphere. Without loss of generality the ϕ coordinate can be defined to be the same as in the embedding 3-dimensional space, but isotropy implies that in dl^2 it should enter only as $r^2 d\phi^2$. This allows us to identify

$$r = \sin\theta, \quad \theta = \sin^{-1} r, \quad d\theta = \frac{dr}{\sqrt{1-r^2}}$$

Then

$$dl^2 = \frac{dr^2}{1-r^2} + r^2 d\phi^2$$

is our desired 2-d metric on a spherical surface. Similarly

$$dl^2 = \frac{dr^2}{1+r^2} + r^2 d\phi^2$$

gives a hyperbolic surface.

Extending this to a 3-dimensional surface embedded in a 4-dimensional hyperspace we get the 3-d spatial metric

$$dl^2 = \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

where $k = 0$ corresponds to flat, $k = 1$ to spherical (closed) and $k = -1$ to hyperboloidal (open) space. The full metric to describe the homogeneous, isotropic universe can then be written as

$$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right]$$

This is the Friedman-Robertson-Walker (FRW) metric for the Universe used in Cosmology.

The metric coefficients involve only a and k, which are determined, via Einstein's equation, from the matter-energy content of the universe.

The scale factor

$$a(t) = a(t, \rho(t), P(t), \Lambda)$$

describes the "expansion" of the Universe or "Hubble flow". The "Hubble Constant" is defined as

$$H_0 \equiv \frac{\dot{a}}{a}(t_0) \approx 65 \ \ \mathrm{km\ s^{-1}\ Mpc^{-1}}$$

where t_0 is the present epoch. The reciprocal of the Hubble constant, called the "Hubble Time", gives approximately the age of the Universe.