The Isothermal Sphere

For a polytropic equation of state

$$P = K\rho^{\gamma} \equiv K\rho^{1+\frac{1}{n}}$$

we have the hydrostatic balance equation

$$\frac{d\Phi}{dr} = -\gamma K \rho^{\gamma-2}\frac{d\rho}{dr}$$

where Φ is the gravitational potential.

In case of an "isothermal" equation of state $P \propto \rho$, i.e., $\gamma = 1$ and $n = \infty$. The hydrostatic equation can then be integrated to

$$\Phi = -K \ln\left(\frac{\rho}{\rho_{\rm c}}\right)$$

where Φ has been assumed to be zero at the centre. Hence

$$\rho = \rho_{\rm c} \exp\left(-\frac{\Phi}{K}\right)$$

Inserting this in the Poisson's equation one finds

$$\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} = 4\pi G\rho_{\rm c} \exp\left(-\frac{\Phi}{K}\right) \tag{1}$$

or an equivalent form in terms of density:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}(\ln\rho)\right) = -\frac{4\pi G\rho}{K} \tag{2}$$

which has an exact solution

$$\rho(r) = \frac{K}{2\pi G r^2} \tag{3}$$

In this case the density is not finite at the centre, so such configurations are called "Singular Isothermal Spheres" (SIS).

Non-singular solutions can be determined by imposing the appropriate boundary conditions. Writing eq. (1) as

$$\frac{d^2w}{dz^2} + \frac{2}{z}\frac{dw}{dz} = \exp(-w) \tag{4}$$

in terms of the dimensionless variables

$$z = Ar, \quad A^2 = \frac{4\pi G\rho_c}{K}, \quad \Phi = Kw,$$

one could impose the boundary conditions

$$w = w' = 0 \text{ at } z = 0$$

and solve the equation numerically to yield a non-singular solution. Eq (4) is referred to as the "Isothermal Lane Emden equation". From the boundary conditions it is apparent that near the centre ($z = 0$), w behaves as $w \propto z^2$, and hence the density behaves as

$$\rho = \frac{\rho_c}{\exp(w)} \approx \frac{\rho_c}{1 + (z/z_0)^2} = \frac{\rho_c}{1 + (r/r_0)^2}$$

where the scaling radius

$$r_0 \equiv \frac{z_0}{A} = \sqrt{\frac{9K}{4\pi G\rho_c}}$$

is known as the "core radius" or the "King radius". Note that the SIS behaviour ($\rho \propto r^{-2}$) is recovered at $r \gg r_0$.

The isothermal sphere is a configuration of infinite radius, as expected for $n > 5$, and infinite mass. It is easy to see from the SIS density profile that the mass included within a radius r grows as

$$M(r) = \int_0^r 4\pi r^2 \rho(r) dr = \frac{2\pi K}{G} r$$

i.e., linearly with radius.

Where isothermal sphere models are applicable in practice, for example, stellar cores with no nuclear burning, or star clusters, external circumstances, such as non-zero surface pressure or tidal interaction with nearby bodies, truncate the solution to a finite radius and mass.