Statistics of Gas and Radiation

Most of the matter in the Universe exists in gaseous form. A fraction ($\sim$ 10%) of it is baryonic matter, like what we are familiar with in everyday experience, and a larger fraction ($\sim$ 90%) is non-baryonic "dark matter" the nature of which we are not very sure about. The "dark matter" makes itself felt only through its gravitational interaction. All "visible" matter is therefore still baryonic.

The baryonic matter in the universe is composed mainly of Hydrogen (about three-quarters of the mass) and Helium (about a quarter). There is a small fraction of heavier elements which, collectively, are referred to as "metals". The abundance of "metals" in the solar neighbourhood is about 2% by mass, while the Hydrogen abundance is $\sim$ 71%. Table 1 shows the mass fraction of several elements in the solar neighbourhood.

Table 1: Mass fraction of different elements in the Solar neighbourhood

H	0.71E+00
He	0.28E+00
C	0.34E-02
N	0.99E-03
O	0.96E-02
Ne	0.18E-02
Na	0.35E-04
Mg	0.66E-03
Al	0.56E-04
Si	0.70E-03
S	0.37E-03
Cl	0.47E-05
Ar	0.11E-03
Ca	0.65E-04
Cr	0.18E-04
Fe	0.13E-02
Co	0.36E-05
Ni	0.73E-04

Equation of State

The equation of state of a gas represents a relation connecting the state variables such as pressure, density, temperature, internal energy etc. The relation between pressure and other state variables will be of much interest, so let us first examine this. We will deal with gas where the energy of mutual interaction between particles is negligible.

A gas will have particles with a distribution of kinetic energies of random motion, and a kinetic energy ϵ would be associated with a momentum of the particle, where the magnitude of the momentum $p(\epsilon)$ is determined by the energy. The pressure is defined as the rate of momentum transfer in a given direction through an unit area per unit time, and since the direction of the momentum is randomly distributed in 3 dimensions, the pressure P is given by

$$P = \frac{1}{3}\int_0^\infty n(\epsilon)p(\epsilon)\mathrm{v}(\epsilon)d\epsilon$$

where $n(\epsilon)d\epsilon$ is the number of particles with energy between ϵ and $\epsilon + d\epsilon$ in an unit volume of the gas, and $\mathrm{v}(\epsilon)$ is the speed associated with the kinetic energy ϵ.

We assume that particles of the gas in question have no internal degrees of freedom, and the kinetic energy ϵ is entirely that of their random translational motion.

If the random motion in the gas is non-relativistic, then we can write $\mathrm{v} = p/m$ and $\epsilon = p^2/2m$, resulting in

$$P = \frac{2}{3}\int_0^\infty n(\epsilon)\epsilon d\epsilon = \frac{2}{3}u_k$$

where u_k is the internal energy density (kinetic energy of random motion per unit volume).

If the gas is relativistic, $\mathrm{v} = c$ and $\epsilon = pc$, giving

$$P = \frac{1}{3}\int_0^\infty n(\epsilon)\epsilon d\epsilon = \frac{1}{3}u_k$$

The proportionality constant between P and u_k has a physical significance: it is equal to $\gamma - 1$, where γ, called the "adiabatic index", equals the ratio of specific heats for a thermal gas. Reversible adiabatic processes yield PV^γ =constant, where V is the volume of the gas. Using this, one finds that for a spherical adiabatic expansion the total internal energy $U_k = u_k V$ of a gas (we use the notation U_k for the total internal energy and U for the total energy including rest energy) drops as R^{-2} for a non-relativistic gas and as R^{-1} for a relativistic gas ($V \propto R^3$).

We will be particularly interested in the relation between pressure P, mass density ρ and temperature T of the gas. This, obviously, is determined by the dependence of u_k on these quantities.

At this point let us remind ourselves what we mean by a "thermal gas". For a classical gas, this means that all energy levels of the gas, both discrete and continuous, are occupied according to the Boltzmann distribution:

$$N(\epsilon) \propto g(\epsilon) e^{-\epsilon/kT}$$

where $g(\epsilon)$ is the so-called "density of states". In case of quantum statistics, the corresponding distribution is:

$$N(\epsilon) \propto \frac{g(\epsilon)}{e^{(\epsilon-\mu)/kT} \pm 1}$$

where the positive sign in the denominator corresponds to a Fermi gas and the negative sign to a Bose gas. Quantum statistics comes into play only when the number of particles per phase space cell of volume h^3 is of order unity. For dilute gases, as encountered in most astrophysical situations, classical description is quite adequate.

If the energy distribution of particles do not follow the above laws, we call such a distribution "non-thermal". In many astrophysical situations we come across "power-law" distribution of particle energies:

$$N(\epsilon) \propto \epsilon^{-\mathrm{p}}$$

which are examples of non-thermal distribution. We will discuss later how such energy distributions are produced.

Quite often the gases encountered in astrophysics do have thermal distribution, but with small departures. For example, in an atomic hydrogen gas the random motion of the atoms may be describable by a Boltzmann distribution at a certain temperature $T_{\rm k}$, while the population in the excited levels of the atoms may not follow the ratio predicted by the same Boltzmann distribution. While strictly speaking this situation is "non-thermal", we still like to describe such a gas as a "thermal gas", but with a difference between the "excitation temperature" $T_{\rm ex}$ for the atomic level in question and the "kinetic temperature" $T_{\rm k}$. In general, $T_{\rm ex}$ may be different for different energy levels.

For a classical thermal gas at a temperature T the average energy (of random translational motion) per particle is $\langle\epsilon\rangle = 3kT/2$ in non-relativistic case and $\langle\epsilon\rangle = 3kT$ in relativistic case. The corresponding values of energy density u_k are then $3nkT/2$ and $3nkT$, where n is the number density of the gas. These yield the familiar expression

$$P = nkT$$

in both regimes. If the gas is composed of multiple species, for example an ionised hydrogen gas is composed of protons and electrons, then the above relation is true for the partial pressure of each species, and the total pressure is the sum of all the partial pressures.

The relation between pressure and mass density ρ can then be obtained from effective mass per particle. If particles are non-relativistic, this is given by

$$P = \frac{\rho}{\mu m_p} kT$$

where μ is the "mean molecular weight" and m_p is the proton mass. For example, pure atomic hydrogen has $\mu = 1$, a mixture (by mass) of 75% Hydrogen and 25% Helium, both in atomic form, has $\mu = 1.23$ and pure, fully ionised Hydrogen plasma has $\mu = 0.5$. Sometimes we will have to deal with the partial pressure of electrons alone, and then we would have to define a "mean molecular weight per electron" μ_e. For a fully ionised hydrogen plasma μ_e is 1.0.

If the gas is relativistic then the appropriate correction to the mass of a particle is to be taken into account. For ϵ much larger than the rest energy the mass can be written as ϵ/c^2, which means that $\rho = u_k/c^2$ and

$$P = \frac{1}{3}\rho c^2$$

independent of temperature. However there could be an intermediate situation where, say in a mixture of electrons and ions, the average energy per particle is larger then 0.5 MeV, high enough for electrons to go relativistic, but is much less than a GeV, so ions are non-relativistic. In this situation we can still get the partial pressure of the electrons by setting $n_e = \rho/\mu_e m_p$, where n_e is the electron number density.

Let us now look at situations where quantum statistics becomes important. This pertains to matter at high density where the Fermi statistics of electrons, protons or neutrons play a role. For a Fermi gas, Pauli's exclusion principle demands that only one particle can occupy one quantum state. Per unit physical volume, the number of states in phase space between momentum 0 to p is $2\times 4\pi p^3/3h^3$ where the factor 2 comes from spin degeneracy. The value of p for which the number of states becomes equal to the actual number of particles present in that unit volume is called the "Fermi momentum" p_F and the corresponding energy is called the "Fermi Energy" ϵ_F. This Fermi energy plays the role of the chemical potential μ in the expression for Fermi-Dirac distribution. Clearly,

$$p_F = \left(\frac{3h^3}{8\pi}\right)^{1/3} n^{1/3}$$

where n is the number density of the species in question. The Fermi energy

$$\epsilon_F = \frac{p_F^2}{2m} \propto n^{2/3} \quad \text{(non-relativistic)}$$

or

$$\epsilon_F = cp_F \propto n^{1/3} \quad \text{(relativistic)}$$

m being the rest mass of a particle.

The quantum statistics of the gas will manifest itself once $\mu (= \epsilon_F)$ becomes of the order of kT or higher. This is the case in the interior of compact stars such as white dwarfs or neutron stars. Such a gas is called "degenerate". In the limit $T \to 0$ the pressure does not vanish. This is called the "degeneracy pressure".

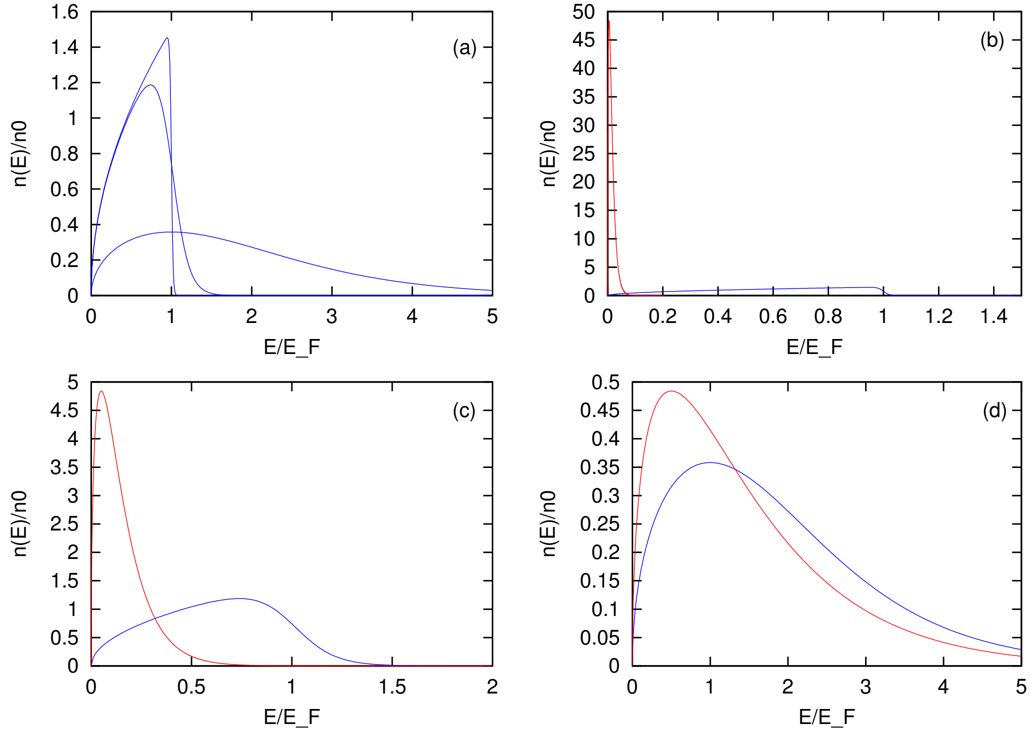

Figure 1: Fermi distributions for translational degrees of freedom in 3 dimensions, for three different values of ϵ_F/kT: 0.01, 0.1 and 1.0 (blue lines). Panel (a) compares these three distributions, and the other three panels compares them with the Boltzmann distribution at the corresponding temperatures (red lines): (b) $kT = 0.01\epsilon_F$, (c) $kT = 0.1\epsilon_F$ and (d) $kT = \epsilon_F$.

Figure 1 shows a comparison between Fermi distribution at different temperatures and their Boltzmann counterparts.

At $kT \ll \epsilon_F$, the gas behaves essentially as a zero temperature gas, with states up to the Fermi level all filled. The energy density u_k in this configuration is then proportional to $n\epsilon_F$, and hence

$$P \propto n^{5/3} \quad \text{(non-relativistic)}$$

$$P \propto n^{4/3} \quad \text{(relativistic)}$$

As above, one can express n in terms of the mass density to get

$$P \propto \left(\frac{\rho}{\mu m_p}\right)^{5/3} \qquad \text{(non-relativistic)}$$

If the particles that are relativistic are also the main source of mass, then according to the general result above $P \propto \rho$, but in the case where degeneracy pressure comes from relativistic electrons and the mass from non-relativistic protons, one gets

$$P_e \propto \left(\frac{\rho}{\mu_e m_p}\right)^{4/3} \qquad \text{(relativistic)}$$

The thermal distribution of radiation is described by the Blackbody function

$$u_\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT} - 1}$$

which gives the spectral energy density at a frequency ν. The corresponding specific intensity (energy flowing per unit area per unit time per unit solid angle per unit frequency interval) at that frequency is given by $I_\nu = cu_\nu/4\pi$. Integration over frequencies gives $u = aT^4$, where a is the radiation constant. Total flux crossing an unit area similarly works out to be σT^4 where $\sigma = ac/4$ is the Stefan's constant. Radiation exerts pressure too, and since photons are massless, the situation is always relativistic. Hence the pressure of thermal radiation is given by

$$P_{\rm rad} = \frac{1}{3}aT^4$$

Unlike in the cases of massive particles discussed above, the energy density here goes as fourth power of temperature since, although the energy per photon remains $\sim kT$, the number density of photons goes as T^3. This is because being particles of zero rest mass, photons can be created and destroyed freely. As seen in figure 2, this makes the black body curve at a higher temperature completely envelope that at a lower temperature. In the case of massive particles the thermal distribution at a higher temperature would intersect that at a lower temperature to ensure the conservation of particle number. In other words, the blackbody distribution is a one-parameter family of curves, determined solely by temperature, while the thermal distribution of massive particles is a two-parameter family, determined both by temperature and particle number.

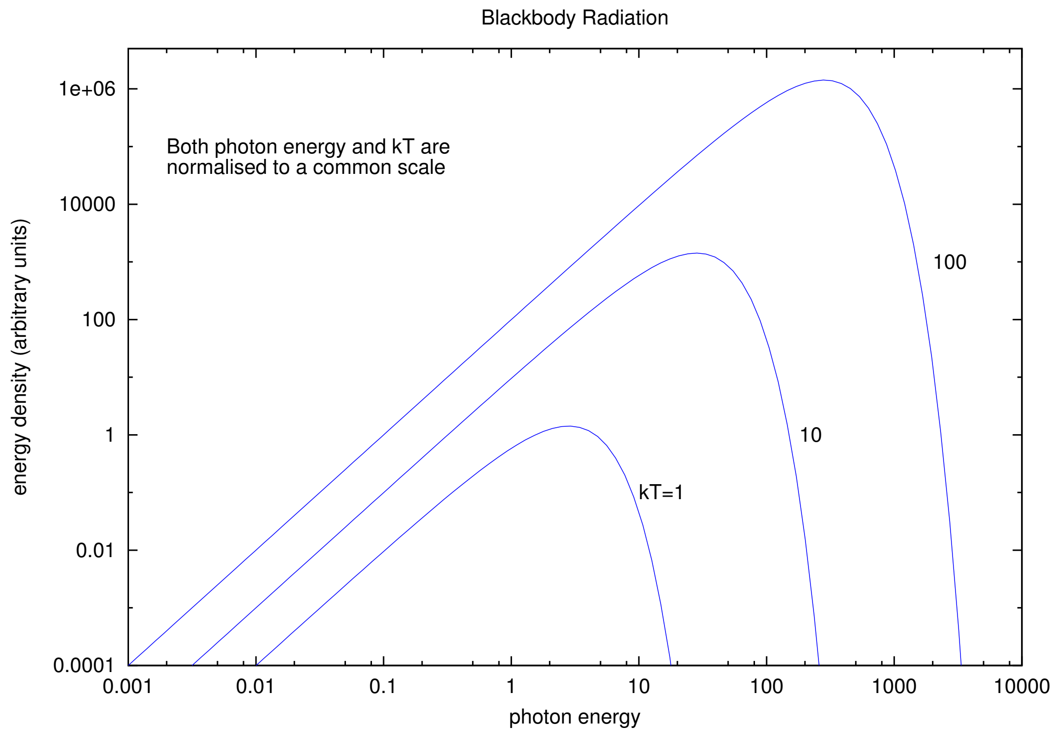

Figure 2: Distribution of energy density u_ν of Blackbody Radiation shown for three different temperatures.

In the special case of the distribution of massive particles where kT is much larger than the rest energy of the particles, and the distribution is in thermal equilibrium with radiation at the same temperature, particles and antiparticles of this species can be created and destroyed freely, and even for these massive particles then the energy density u would go as T^4.