Stellar Dynamics: Collisionless systems

Let us consider a collisionless stellar system with N stars of mass m each, so that the total mass of the system is $M = Nm$.

In the limit of a very large N the equation of motion of any star can be written in terms of a smooth mean-field gravitational potential $\phi(\vec{r}, t)$:

$$\ddot{\vec{r}} = \vec{\nabla}\phi \tag{1}$$

$$\phi(\vec{r}, t) = -G \int \frac{\rho(\vec{r}', t)}{|\vec{r} - \vec{r}'|} d^3r' \tag{2}$$

The distribution of stars is described by a phase space distribution function

$$f(\vec{r}, \vec{v}, t) d^3r\, d^3v = \text{ no. of stars in phase space volume } d^3r\, d^3v$$

A similar description will also apply to Dark Matter particles associated with the stellar system.

The number density is then

$$n(\vec{r}, t) = \int f(\vec{r}, \vec{v}, t) d^3v \tag{3}$$

And the mass density

$$\rho(\vec{r}, t) = \int m f(\vec{r}, \vec{v}, t) d^3v \tag{4}$$

The motion of a star in the system is dictated by a Hamiltonian

$$H = \frac{v^2}{2} + \phi(\vec{r}, t) \tag{5}$$

which equals the energy per unit mass of any star or test particle. The flow in phase space therefore obeys Liouville's theorem, that the phase space

$d^3r\,d^3v$ occupied by any given set of matter is conserved. In conjunction with mass conservation, this yields that

$$f(\vec{r},\vec{v},t)d^3r\,d^3v = \text{ constant along phase trajectories}$$

The above set of equations have to be solved self-consistently to obtain the dynamics of the system.

This is often re-cast in the form of the collisionless Boltzmann equation

$$\frac{\partial f(\vec{r},\vec{v},t)}{\partial t} + \vec{v}.\nabla f - \nabla\phi.\nabla_v f = 0 \tag{6}$$

with

$$\phi(\vec{r},t) = -G\int \frac{m f(\vec{r}',\vec{v}',t)}{|\vec{r}-\vec{r}'|} d^3r' d^3v' \tag{7}$$

The second relation can be expressed equivalently as a Poisson's equation

$$\nabla^2\phi(\vec{r}',t) = 4\pi G m \int f(\vec{r},\vec{v},t)d^3v \tag{8}$$

with appropriate boundary condition imposed at infinity. The dynamics can then be obtained by a self-consistent solution of eq. 6 with either eq. 7 or eq. 8.

Often one is interested in solutions of the Collisionless Boltzmann Equation that describe motion in steady $\phi(\vec{r})$, such that the distribution function is also steady. Such $f(\vec{r},\vec{v})$ can be written in terms of the integrals of motion (IoM) of a star's orbit. The IoM are functions $I(\vec{r},\vec{v})$ of a star's position and velocity that remain constant along the orbit of a given star. The value of the IoM may be different for different stars.

One example of an IoM is the energy per unit mass

$$E(\vec{r},\vec{v}) = \frac{v^2}{2} + \phi(\vec{r}) \tag{9}$$

In an axisymmetric potential $\phi(R,z)$ the z-component of the angular momentum L_z is an IoM. In a spherically symmetric system the total angular momentum $\vec{L}$ is an IoM. If, in a disk, the motion perpendicular to the disk (in z-direction) is independent of rotation in the plane of the disk then

$$E_z = \frac{v_z^2}{2} + \phi(R_0, z)$$

is an IoM.

Once a set of IoM-s are identified, any function f(IoM) would satisfy the Collisionless Boltzmann Equation.

Spherical systems

In the simplest case the distribution function could be considered to be a function only of the orbital energies

$$f = f(E)$$

where E is the energy per unit mass. The self-consistent Poisson Equation then becomes

$$\nabla^2\phi = 4\pi G \int m f(E) d^3v = 4\pi G\rho(\phi) \tag{10}$$

all physical solutions of which are spherically symmetric.

For example, if one assumes a distribution function of the form

$$f(E) = f_0 \exp\left(-\frac{E}{\sigma^2}\right) \tag{11}$$

then one arrives at the Isothermal Sphere solution discussed earlier. With the boundary condition of a finite density ρ_0 at the centre, the same distribution yields a density profile of the form

$$\rho(r) = \frac{\rho_0}{1 + (r/r_c)^2}$$

where r_c is the core radius, as discussed earlier in the context of isothermal spheres.

These isothermal sphere models, however, extend to infinity since the distribution contains stars up to infinite energy. In reality the distribution is truncated at some maximum energy E_0, with the corresponding distribution function being given by

$$\begin{aligned} f(E) &= f_0\left[\exp\left(-\frac{E}{\sigma^2}\right) - \exp\left(-\frac{E_0}{\sigma^2}\right)\right] \quad (E \le E_0) \\ &= 0 \qquad\qquad (E > E_0) \end{aligned}$$

This results in a finite configuration with an outer radius r_T, called the *Tidal Radius*, where $\phi(r_T) = E_0$. These models are called *King Models* after Ivan King who introduced them in 1966 (Astronomical Journal vol. 71, p. 64). The density distribution in a King model can be worked out numerically, and is characterized by a concentration parameter $c = \log(r_T/r_c)$. At $r \ll r_T$ the density distribution is very similar to the isothermal sphere models, but it cuts off sharply as r approaches r_T. King models are routinely used to describe the density distribution in globular star clusters and also in some elliptical galaxies. For a typical globular cluster in our Galaxy r_T/r_c is about 10. For an elliptical galaxy r_T/r_c could exceed 200.

Elliptical galaxies

The surface brightness distribution of an elliptical galaxy is usually well fit by an expression of the form

$$I(R) = I(0)\exp\left(-\frac{R}{R_0}\right)^{1/4} \tag{12}$$

known as de Vaucoulers' law. Here R represents the distance from the centre, along the major axis, in the two-dimensional projected image of the galaxy, and R_0 is a scaling distance. $I(R)$ is obtained by summing the observed flux in an elliptical isophotal ring of major axis R and a small width. Using this, one can obtain the total optical luminosity of an elliptical galaxy even in situations where noise in the image hampers detectability of the faint outer edges of the galaxy. The luminosity L of an elliptical galaxy appears to correlate well with the radial velocity dispersion σ_r^2, which can be measured using the doppler width of spectral lines seen from the galaxy. This correlation, of approximate form $L \propto \sigma_r^4$, is known as the Faber-Jackson relation. A similar, and much tighter correlation exists in disk galaxies between the luminosity and the circular velocity in the flat part of their rotation curves: $L \propto v_{\rm flat}^4$. This is known as the Tully-Fisher relation.

The velocity dispersion in an elliptical galaxy yields an estimate of its "dynamical" or "virial" mass $M_{\rm vir}$. The kinetic energy of the system is

$$T = \frac{3}{2}\sigma_r^2 M_{\rm vir}$$

and the potential energy

$$V = -\frac{GM_{\rm vir}^2}{2r_c}$$

where r_c is the core radius. Virial equilibrium demands that

$$T = -\frac{V}{2}$$

and hence

$$M_{\rm vir} = \frac{6\sigma_r^2 r_c}{G}$$

Disk galaxies

A disk galaxy is an axisymmetric system. The simplest distribution function describing such a system would be a function of at least two IoMs, energy per unit mass E and the z-component of the angular momentum L_z. One may write the effective radial potential for motion in the plane of the disk as

$$\Phi_{\rm eff}(R) = \phi(R, z = 0) + \frac{L_z^2}{2R^2}$$

The motion consists of a rotation around the galactic centre, and a radial motion in the effective potential. A typical form of the effective potential is sketched in fig. 1. The frequency of radial motion will not necessarily be the same as or commensurate with the frequency of rotation around the galactic centre. The orbital motion of a star in the disk plane can be approximately described as epicyclic, a superposition of rotation along the galactic centre with angular frequency Ω and a radial oscillation with an epicyclic frequency κ:

$$R = R_g + X\cos(\kappa t + \psi) \tag{13}$$

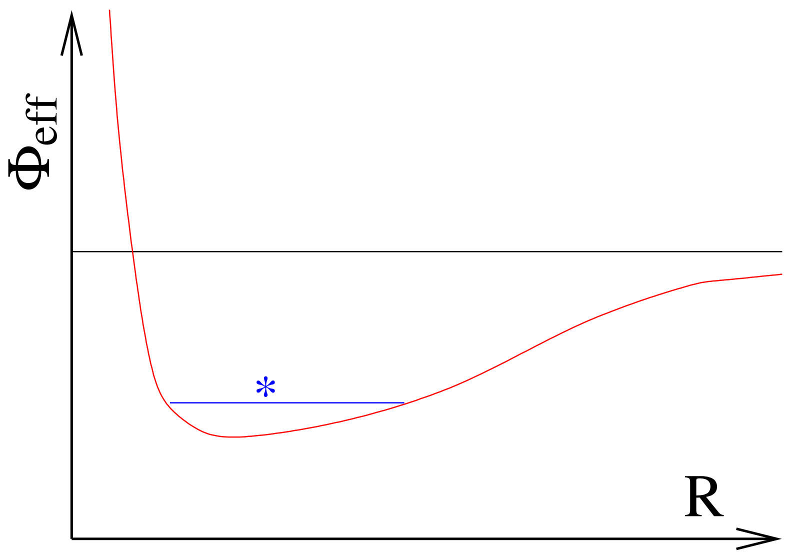

Figure 1: Effective radial potential for motion in the disk of an axisymmetric stellar system (schematic). A star would execute radial oscillations in a bound orbit, as shown here in blue and marked by an asterisk.

where R_g is the average distance of the star from the galactic centre and X is the radial oscillation amplitude. In our galaxy, near the sun, the epicyclic frequency κ is about 1.4Ω.

Most disk galaxies, including ours, show clear spiral arms which contain higher-than-average density of matter. They are also sites of recent star formation and are therefore rich in luminous, massive stars. These spiral arms are patterns in the flow of matter in the disk. Matter moves through the spiral arms, the density enhancement resulting mainly from slightly slower motion in the arms and faster outside the arms. The arms may be described as a "density wave" on the disk. Originally excited perhaps by interaction with neighbouring bodies, the waves are stabilised by the mutual gravitational interaction of stars and gas clouds. A simple kinematic description of the spiral pattern could be as follows. Consider the epicyclic orbits of stars at various distances from the galactic centre. We take an example of eq. 13 with $\kappa = 2\Omega$ in fig. 2. The orbits assume an oval shape in this case. In the upper panel we see all the ovals aligned in the same

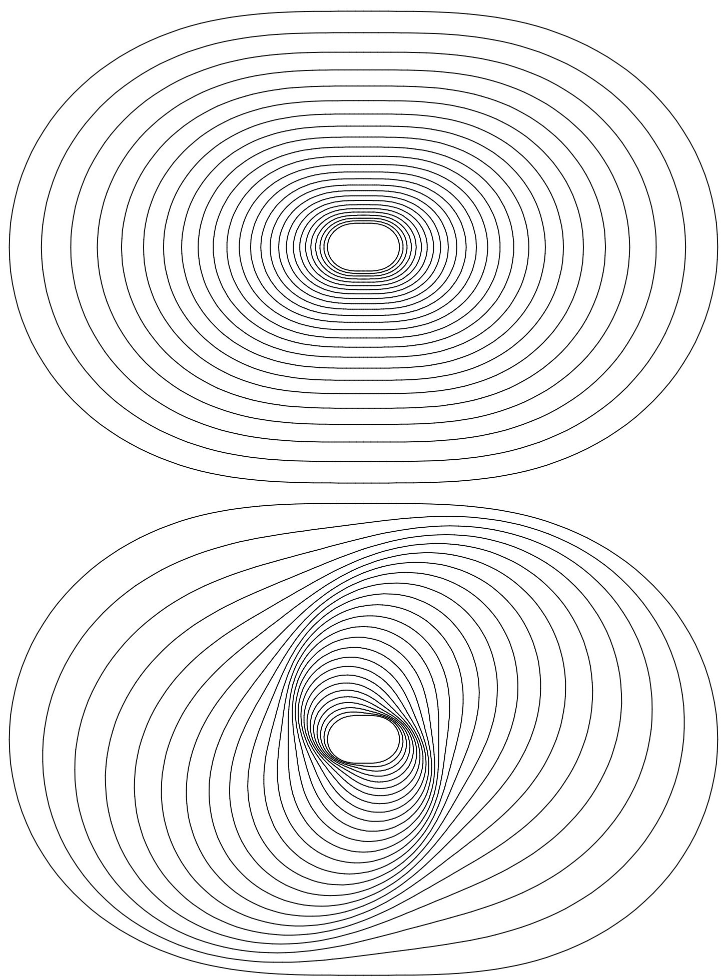

Figure 2: Kinematic spiral in a disk galaxy. The upper panel shows epicyclic orbits from inner to outer reaches of the galaxy, all lined up in the same direction. In the lower panel a spiral pattern is shown to originate if the orbits are progressively tilted with respect to each other

direction, which corresponds to setting ψ the same for all the orbits. In the lower panel we see the orbital configuration if ψ is progressively increased from the inner to the outer orbits. A spiral pattern is clearly evident, and it corresponds to the set of points where the orbits come closest to each other, thus enhancing the density.

It is notable that nearly half of the disk galaxies also show prominent central "bars". Our galaxy, too, has a bar occupying the central regions of the galaxy. Forming a bar is one way of decreasing gravitational energy for a given total angular momentum.

We have described a disk galaxy above with a distribution function depending on two IoMs, E and L_z. The energy per unit mass in an orbit is

$$E = \frac{v_R^2 + v_z^2}{2} + \frac{L_z^2}{2R^2} + \phi(R, z)$$

The symmetry of this suggests that in the distribution the R and the z components of the velocity dispersion $< v_R^2 >$ and $< v_z^2 >$ must be equal. In our galaxy, in the solar neghbourhood, this is however not found to be the case. A proper description of the distribution function should therefore involve a third, independent IoM. The exact nature of this third integral still remains a matter of debate.

The population of galaxies

We have talked about two extreme types of galaxies, ellipticals and highly flattened disks. Galaxies however come in many forms intermediate between the two and have been classified into a morphological sequence originally by Edwin Hubble and later extended by Alan Sandage. Ellipticals are designated with labels E0 to E7 for progressively higher degree of ellipticity. Following this comes S0, also called lenticular galaxies which have a pronounced elliptical central bulge and a relatively faint disk with no discernible spiral pattern. Spiral galaxies then follow, labelled Sa, Sb, Sc, Sd or Sm according to the richness of the spiral pattern. A parallel branch to this contains the barred spirals, designated SBa, SBb etc. Our galaxy, the Milky Way, is classified as type SBc. Some galaxies may have peculiar morphology and defy this classification scheme altogether. They are called Irregular galaxies. One of the close satellites of the Milky Way, the Large Magellanic Cloud (LMC) is an example of an Irregular galaxy.

The distribution of the luminosities of galaxies can be well fitted by a function of the following form, called the Schechter luminosity function:

$$n(x)dx = \phi^* x^a e^{-x} dx \tag{14}$$

where $x \equiv L/L_*$ is the galaxy luminosity normalised to a scale L_* above which the distribution cuts off exponentially. At lower luminosities the luminosity function is a power-law. A fit to field galaxies in the nearby universe yields $a \approx -1$, $\phi^* \approx 1.4 \times 10^{-2} h^3$ Mpc^{-3}. L_* is about 10^{10} Solar luminosities in the local Universe. Here h is the Hubble constant measured in the units of 100 km s^{-1} Mpc^{-1}. The current estimate of h is about 0.7.

Galaxies form a clustered distribution at small scales, the distribution becoming gradually uniform if averaged over larger and larger scales. One way to quantify this is the two-point correlation function of galaxies. Let n be the average spatial density of galaxies, and let us consider two small volumes ΔV_1 and ΔV_2 at a distance r_{12} from each other. The *à priori* probability of finding a galaxy in ΔV_1 is $n\Delta V_1$, and that of finding one in ΔV_2 would be $n\Delta V_2$. If, however, galaxies clump together, then the joint probability of finding one galaxy in ΔV_1 and also another in ΔV_2 will be larger than the product of the *à priori* probabilities, and can be written as:

$$\Delta P = n^2[1 + \xi(r_{12})]\Delta V_1 \Delta V_2$$

The quantity $\xi(r)$ in the above expression is the two-point correlation function. In the galaxy distribution $\xi(r)$ is found to be positive at small distances, less than about 100 Mpc, indicating that over these scales galaxies cluster significantly. The observed correlation function is well described by a power law for $r \leq 50h^{-1}$ Mpc:

$$\xi \approx \left(\frac{r}{r_0}\right)^{-\gamma}; \quad \gamma \approx 1.8, \;\; r_0 \approx 5h^{-1}\,\mathrm{Mpc}$$

About half of all galaxies reside in well-defined clusters or groups. In a typical cluster one finds about 50 to 100 galaxies in the central Mpc. A rich cluster may be several Mpc in size and contain thousands of galaxies. The nearest known cluster is the Virgo Cluster, located 15 to 20 Mpc from us, and containing about 1300 catalogued members. Groups are less dense entities and contain smaller number of galaxies. Our Galaxy, the Milky Way, is a member of a group we call the Local Group, which is about 1 Mpc in radius. The largest member of the Local Group is the Andromeda galaxy (M31), about twice as massive as the Milky Way. It is located 770 kpc away from us, and is approaching the Milky Way with a speed of about 120 km/s. In the Local Group there are at least 30 members within 1 Mpc of the Milky Way, and there may be many undiscovered dwarfs. The Milky Way itself has about 12 smaller, satellite galaxies. Galaxies within ~ 30 Mpc of the Milky Way appear to lie in a flattened disk-like distribution. The mid-plane of this distribution is called the "supergalactic plane", which is nearly perpendicular to the disk of the Milky Way.

In Groups, the dominant species of galaxies tend to be spirals (there are Groups known to consist of spirals alone), while in clusters they happen to be ellipticals. In the densest parts of the groups and clusters the fraction of ellipticals rises. Even the spirals found in the dense regions of groups and clusters are found to be severely stripped of gas. Encounters with other galaxies and the material between galaxies appear to be responsible for removing gas from the galaxies. Diffuse gas in a cluster (the intracluster medium) flows into the centre of the cluster, falling into the gravitational potential well. As a result this gas gets heated up and emits copious amount of X-rays through which it can be observed. Near the centre of rich clusters the gas becomes dense and its cooling time is estimated to be shorter than the Hubble time. This gas may therefore cool and settle, allowing more gas from the outer regions to flow into the centre. This is called a "cooling flow". Attempts to detect these cooling flows are on with the currently active X-ray observatories Chandra and XMM-Newton. The results suggest that cooling flows are not as ubiquitous as they were once thought to be, and additional heating processes may be needed to explain the state of the hot cluster gas as observed.