Resolution of Big Bang Singularity in Loop Quantum Cosmology

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- Universe at low curvature is extremely well described by the Friedman dynamics. However, GR is inadequate at high curvatures.

  Evolve the Universe backwards: For $a \rightarrow 0$, energy density and curvature $\propto a^n$ (n < 0) $\rightarrow \infty$.

  $\Rightarrow$ Big Bang singularity. Evolution Stops.
  Result of powerful singularity theorems.[a]

  Classical GR fails to describe the birth of our Universe. Need of new physics.

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[a] Penrose, Hawking (1960's)

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- Example from Quantum Theory:
  – Rutherford's model of Atom is unstable.

  – Bohr's model: Energy levels discrete. Finite minimum energy $E_{min} = -(me^4/2\hbar^2)$. As $\hbar \to 0$, $E_{min} \to -\infty$.

  Can a quantum theory of gravity resolve the Big Bang singularity?
  Is there any analog of Raichaudhuri equation for the resolution of singularities?

Penrose, Hawking (1960's)

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- However, we have few candidates and from simple models there are some useful insights.
- Some of the questions a quantum theory of cosmology must answer:
  – What is the nature of spacetime at high curvatures/Planck scale?
  – Is there a non-singular origin of the Universe?
  – Is Universe classical or foamy ‘beyond the Big Bang’?
  – Does the non-singular quantum Universe become classical at low curvatures?
  – How do we test the theory?

Some Very Interesting Ideas

Quantum Foam:

- Gravity + Quantum $\rightsquigarrow$ spacetime subject to uncertainty relation. Geometry and its rate of change can not be simultaneously known to an arbitrary precision.[a]
- Quantum fluctuations of the conformal degrees of freedom may resolve the singularity.[b]

The Universe is classical on the other side:

- Pre Big Bang Models: Based on ideas of perturbative string theory (scale factor duality: $a \to 1/a$).[c]
- Ekpyrotic/Cyclic Models: Universe on a brane in a higher dimensional bulk. Big bang a collision between two branes. Cosmic structures originated in the pre big bang phase.[d]

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[a] Wheeler (50's)

[b] Narlikar, Padmanabhan (Late 70's)

[c] Gasperini, Veneziano, ... (90's)

[d] Steinhardt, Turok, Khoury, ... (2001-...)

Our Strategy

Use techniques of Loop Quantum Gravity in Cosmology

Outline:

– Wheeler-DeWitt Quantum Cosmology

– Loop Quantum Cosmology

– Massless Scalar Model: Quantization and Numerical Results

– Exactly Solvable LQC: Robustness of results

– Summary and Outlook

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Quantum Cosmological Models

Based on Metric based canonical (Hamiltonian) quantization.[a]

– Basic variables: Metric g_{ab}, Momentum p^{ab}

– Dynamics obtained from solving constraints and finding equations of motion for observables.

– Hamiltonian constraint non-polynomial, difficult to quantize.

Simplifications for cosmological models (only finitely many degrees of freedom).
$\rightarrow$ Standard quantum mechanical quantization possible.

Geometry $\rightarrow a, p_a(\propto \dot{a}(t))$, Matter $\rightarrow \phi, p_\phi$.

Quantum States: $\Psi(a,\phi)$, $\hat{a}\,\Psi(a,\phi) = a\Psi(a,\phi)$, ...

Hamiltonian:

$$p_a^2 a^2 = const.\mathcal{H}_\phi$$

[a] Misner, Wheeler, DeWitt (70's)

Example: Massless Scalar Field

$$\mathcal{H}_\phi = \frac{p_\phi^2}{2a^3}, \;\; p_\phi = \text{const.}, \;\; \phi \sim \log v \;\; (v \propto a^3).$$

Classically $\rho \sim a^{-6}$. As $a \to 0$, energy density and curvature become infinite.

Wheeler-DeWitt quantization:

Quantum constraint: $\hat{p}_a^2 \hat{a}^2 \, \Psi(a, \phi) = \text{const.} \, \mathcal{H}_\phi \, \Psi(a, \phi)$

leads to the WDW Equation

$$\frac{\partial^2}{\partial \alpha^2} \, \Psi(\alpha, \phi) = \frac{\partial^2}{\partial \phi^2} \, \Psi(\alpha, \phi), \qquad \alpha = \log a$$

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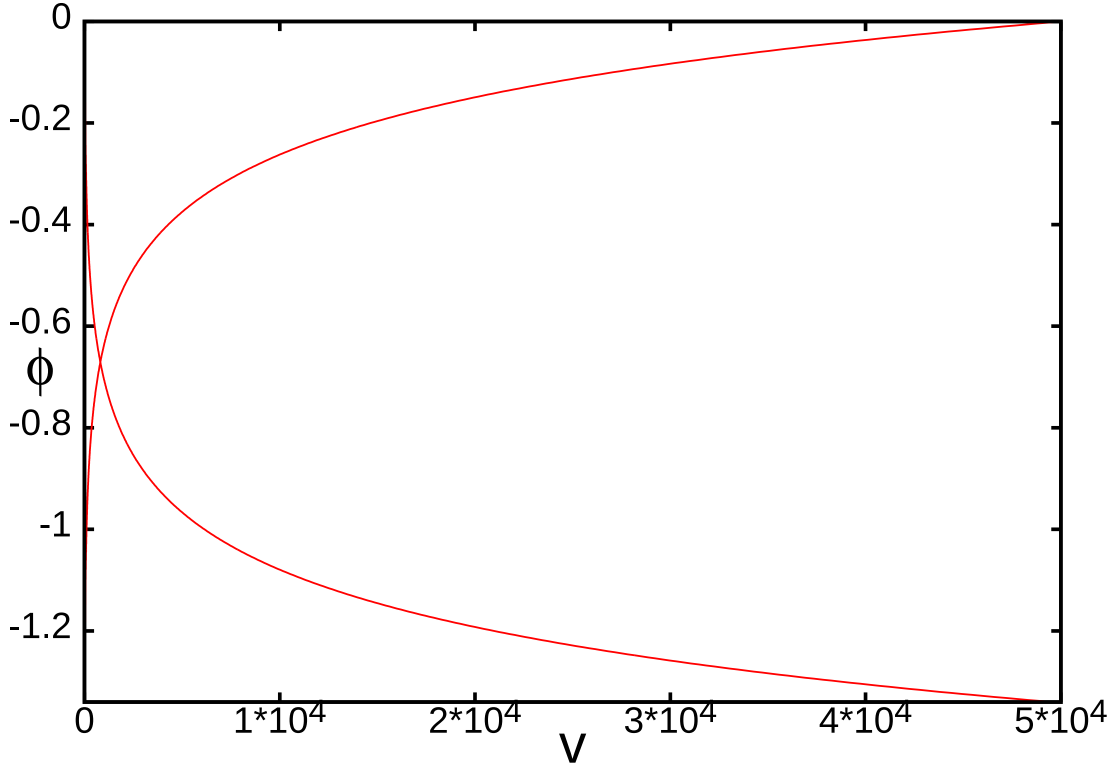

All classical solutions are singular.

Consider semi-classical states peaked at late epoch, evolve backwards towards Big Bang:

Wheeler-DeWitt states follow the classical trajectories into the big bang.

What went wrong?

– A straight forward union of quantum theory and gravity may not work. Naive implementation as a Schrodinger quantum mechanical system fails.

– Spacetime picture essentially the same as in the classical theory.

– No guidance from a full theory of Quantum Gravity.

Loop Quantum Gravity

Based on Ashtekar variables. Gravity casted as a gauge theory.[a]

New phase space variables:

– Connection A^i_a: Matrix valued vector potential (encodes time derivative of spatial metric)

– Triad E^a_i: Three orthonormal vectors (encode metric). Analogous to Electric field.

Enormous simplification of the Hamiltonian constraint: $\mathcal{H} = \epsilon_{ijk} \, E^{ai} E^{bj} \, F^k_{ab}$

Elementary variables:

- Holonomies of connection along a curve: $h(A)$ (Fundamental excitations of quantum geometry)
- Flux across surface: $F(E)$

No operator corresponding to the connection (all classical functions casted in holonomies and fluxes and then quantized).

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[a] Ashtekar (86)

Key Features of the Quantum Theory:

- Based on the Einsteinian philosophy: Spacetime not an inert stage, Is Dynamical. Gravity $\sim$ Dynamics of Spacetime. Quantization of dynamical spacetime.

- Non-perturbative and background independent. Matter and Geometry quantum mechanical from the beginning.

- Background independent QFT.[a] Unique kinematical representation.[b]

- Geometrical operators have discrete spectra.[c]

- Black hole entropy[d e f g h]

- Graviton propagator at low energies.[i]

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[a] Ashtekar, Baez, Isham, Jacobson, Lewandowski, Marolf, Rovelli, Smolin, Thiemann, ... (Mid 90's)

[b] Lewandowski, Sahlmann, Okolow, Thiemann (04); Fleishhack (05)

[c] Ashtekar, Lewandowski; Rovelli, Smolin (Mid 90's)

[d] Ashtekar, Baez, Corichi, Krasnov (98)

[e] Kaul, Majumdar (00)

[f] Domagala, Lewandowski (04); Meissner (04)

[g] Ghosh, Mitra (05)

[h] Corichi, Diaz-Polo, Fernandez-Borja (07); Sahlmann (07)

[i] Engle, Freidel, Krasnov, Livine, Modesto, Rovelli, Speziale, ... (06-...)

LQC: Homogeneous and Isotropic setting

Spatial homogeneity and isotropy: fix a fiducial triad $\mathring{e}_i^a$ and co-triad $\mathring{\omega}_a^i$.
Symmetries $\Rightarrow$

$$A_a^i = c\,\mathring{V}^{-1/3}\mathring{\omega}_a^i, \;\; E_i^a = p\,\mathring{V}^{-2/3}\,(\det\mathring{\omega})\,\mathring{e}_i^a$$

Basic variables: c and p satisfy $\{c, p\} = 8\pi G\gamma/3$.

– Relation to scale factor:
$|p| = a^2$ (two possible orientations for the triad)
$c = \gamma\dot{a}$ (on the space of solutions of GR).

Elementary variables
– Holonomies: $h_k(\mu) = \cos(\mu c/2)\mathbb{I} + 2\sin(\mu c/2)\tau_k$, $\mu \in (-\infty, \infty)$.

Elements of form $\exp(i\mu c/2)$ – generate algebra of almost periodic functions

Hilbert space: $\mathcal{H}_{\rm kin} = L^2(\mathbb{R}_{\rm B}, d\mu)$

Orthonormal basis: $N(\mu) = \exp(i\mu c/2)$; $\langle N(\mu)|N(\mu')\rangle = \delta_{\mu,\mu'}$

- Even at the kinematical level Hilbert space of LQC is different from the Wheeler-DeWitt theory.

  Quantum Mechanics of the Universe in a new representation (in-equivalent to Schroedinger-WDW representation).

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- Question: Why loop quantization not equivalent to Wheeler-DeWitt quantization? (What about Stone-von Neumann Uniqueness Theorem?)

- Even at the kinematical level Hilbert space of LQC is different from the Wheeler-DeWitt theory.

  Quantum Mechanics of the Universe in a new representation (in-equivalent to Schroedinger-WDW representation).

- Question: Why loop quantization not equivalent to Wheeler-DeWitt quantization? (What about Stone-von Neumann Uniqueness Theorem?)

- Answer: No. $\widehat{\exp(i\mu c)}$ well defined but $\hat{c}$ is not $\Rightarrow$ Underlying assumption of Stone-von Neumann Uniqueness theorem violated

Hamiltonian Constraint

$$C_{\rm grav} = - \int_{\mathcal{V}} d^3x \, N \, \varepsilon_{ijk} \, F^i_{ab} \, (E^{aj} E^{bk} / \sqrt{|\det E|})$$

Procedure: Express $C_{\rm grav}$ in terms of elementary variables and their Poisson brackets

– Classical identity of the phase space:[a]

$$\varepsilon_{ijk}(E^{aj} E^{bk}/\sqrt{|\det E|}) \longrightarrow {\rm Tr}(h_k^{(\mu)} \{h_k^{(\mu)-1}, V\} \tau_i)$$

– Express field strength in terms of holonomies: $F^i_{ab} \longrightarrow$ Limit of the holonomy around a loop divided by the area of the loop, as area shrinks to zero.

Area goes to the minimum in quantum theory: $\Delta = \lambda^2$.

Leads to two types of quantum modifications:

(i) Curvature modifications from field strength

(ii) Inverse triad corrections (also for the matter part). Not tied to a curvature scale

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[a] Thiemann (98)

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Quantum constraint (in the $v(= p^{3/2})$ representation): [a]

$$\hat{C}_{\rm grav}\Psi(v) = f_+(v)\Psi(v+4) + f_o(v)\Psi(v) + f_-(v)\Psi(v-4) = \hat{C}_{\rm matt}\Psi(v)$$

Features:

– Difference equation in constant steps of eigenvalues of the volume operator.

– Non-singular for all states.

– $\hat{C}_{\rm grav} \longrightarrow \hat{C}_{\rm grav}^{\rm WDW}$ with natural factor ordering for $|v| \gg 1$.

– Early quantization led to evolution in uniform steps in p.[b] However, on closer inspection theory does not lead to classical GR, and suffers from dependence on $\mathring{V}$.

– Many phenomenologically interesting applications based on inverse triad modifications. [c] [d]

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[a] Ashtekar, Pawlowski, PS (06)

[b] Bojowald (01); Ashtekar, Bojowald, Lewandowski (03)

[c] Inflation: Bojowald, Vandersloot (03); Tsujikawa, PS, Maartens (03); Date, Hossain (04), ...

[d] BHs & Grav. Collapse: Bojowald, Goswami, Maartens, PS (05); Goswami, Joshi, PS (05); Husain, Winkler (05)

What is the physics of singularity resolution ?

- Isolate a 'time' variable.
- Find physical states, physical Hilbert space, inner product and suitable (Dirac) observables.
- Construct semi-classical states at late 'times'.
- Evolve the states backward using quantum Hamiltonian constraint equation.
- Compare with the classical trajectory.

Questions answered for simple models.[a] [b] [c] [d] [e] [f]

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[a] Massless Scalar in Flat Universe (with and without Λ): Ashtekar, Pawlowski, PS (06)

[b] Closed Universe: Ashtekar, Pawlowski, PS, Vandersloot (06)

[c] Open Universe: Vandersloot (07)

[d] Bianchi-I Model (Effective theory understood): Chiou, Vandersloot (07)

[e] Massive Scalar (Inflationary potential): Ashtekar, Pawlowski, PS (08)

[f] Black Hole spacetimes: Ashtekar, Bojowald (05); Boehmer, Vandersloot (07); Campiglia Gambini, Pullin (07)

Massless Scalar Model[a]

Phase space: (c, p, ϕ, p_ϕ), $\{\phi, p_\phi\} = 1$

$$C_{\rm grav} + C_{\rm matt} = -6\frac{c^2}{\gamma^2}\sqrt{|p|} + 8\pi G\,\frac{p_\phi^2}{|p|^{3/2}} \approx 0, \quad p_\phi = {\rm const}, \quad \phi \sim \log v$$

– ϕ is a monotonic function, plays the role of internal time
– Evolution refers to relational dynamics – the way geometry changes with 'time' (ϕ)
– Dirac Observables: p_ϕ, $|v|_\phi$

Quantum constraint: $\partial_\phi^2 \Psi(v, \phi) = -\Theta\Psi(v, \phi)$

$$\Theta\Psi(v,\phi) := \Big[C^+(v)\Psi(v+4,\phi) + C^o(v)\Psi(v,\phi) + C^-(v)\Psi(v-4,\phi)\Big]$$

Constraint similar to the massless Klein-Gordon equation in static spacetime.
$\Theta \to$ Laplacian-type operator (Is self-adjoint and positive definite).

Hilbert space can be constructed as in Klein-Gordon theory (Positive frequency solutions). Physical states satisfy symmetry in v.

Inner product found by (i) group averaging, (ii) demanding the self-adjoint action of observables.

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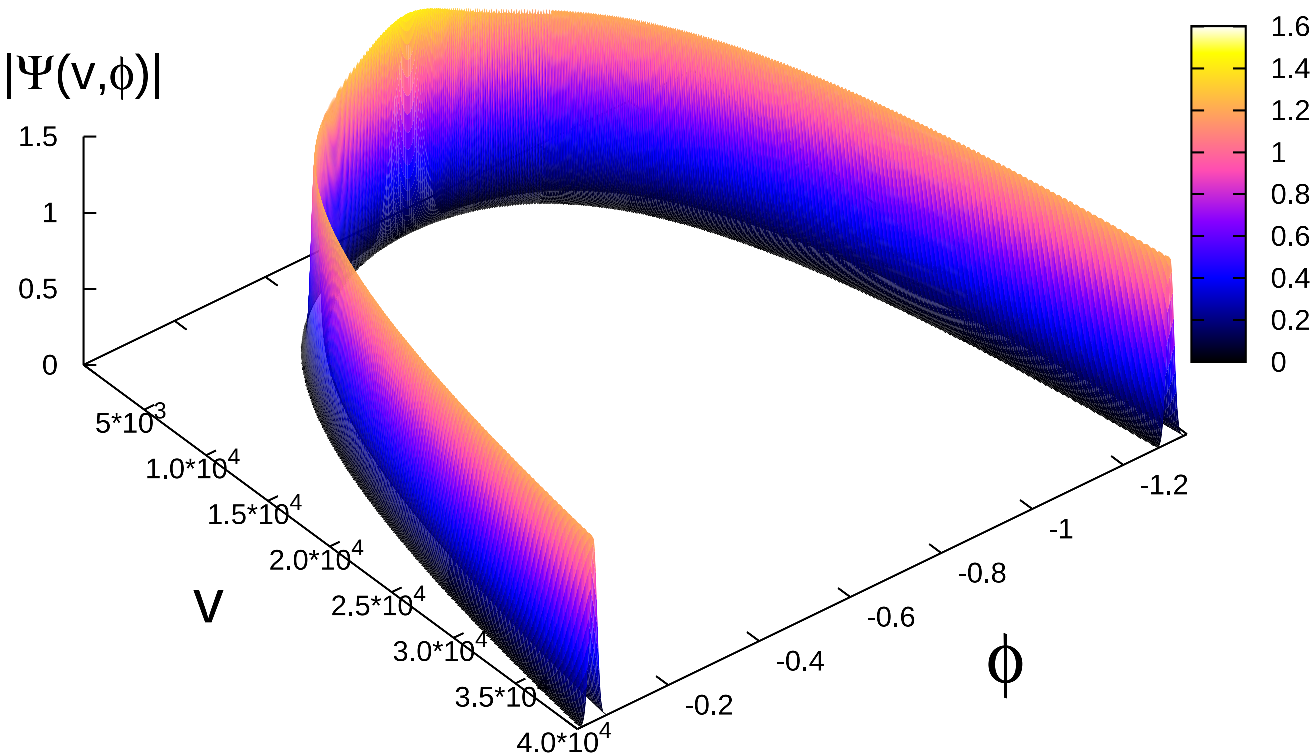
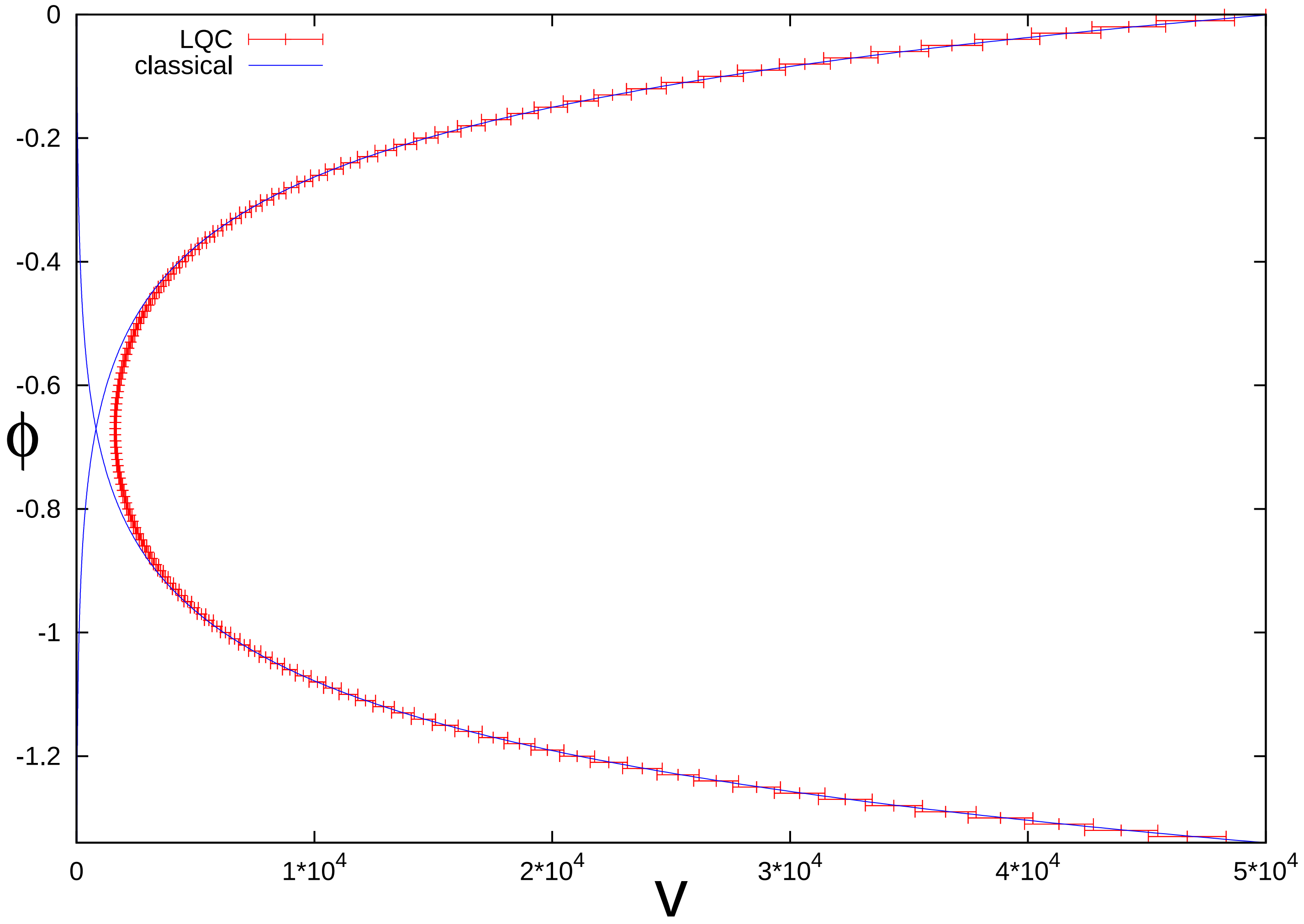
[a] Ashtekar, Pawlowski, PS (06)

Result: Quantum Bounce



Comparison of Evolution



Results of Quantum Evolution (Numerical Simulations)

- States remain sharply peaked through out the evolution. Negligible change in symmetry of the states.
- Expectation values of $v_{|\phi}$ and p_ϕ are in good agreement with classical trajectories until energy density becomes of the order of a critical density $\rho_{\rm crit}$ ($\sim$ 0.82 ρ_{Pl})
- *State does not follow classical trajectory into the Big Bang.* At critical density it bounces from the expanding branch to the contracting branch with same value of $\langle \hat{p}_\phi \rangle$. Big bang replaced by a big bounce at Planck scale.
- Fluctuations of observables remain small.

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Some Features of New Physics:

– Quantum dynamics described by an effective Hamiltonian. Leads to a modified Friedman[a] and Raichaudhuri equation:

$$H^2 = \frac{8\pi G}{3}\,\rho\left(1 - \frac{\rho}{\rho_{\rm crit}}\right)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\,\rho\left(1 - 4\frac{\rho}{\rho_{\rm crit}}\right) - 4\pi G\,P\left(1 - 2\frac{\rho}{\rho_{\rm crit}}\right)$$

– Rich phenomenology.[b] [c] [d] [e] [f] [g] [h] [i]

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[a] Coincidentally also in some braneworld models: Sahni, Shtanov (02)

[b] Cyclic & Pre-Big Bang models: PS, Vandersloot, Vereshchagin (06); De Risi, Maartens, PS (07)

[c] Big Rip avoidance: Sami, PS, Tsujikawa (06)

[d] Scaling solutions: PS (06)

[e] Inflationary models: Zhang, Ling (07); Copeland, Mulryne, Nunes, Shaeri (07)

[f] Tachyon & Quintom Models: Sen (06); Wei, Zhang (07); Xiong, Qiu, Cai, Zhang (07)

[g] Phantom Models: Samart, Gumjudpai (07); Naskar, Ward (07)

[h] Scale invariant thermal fluctuations: Magueijo, PS (07)

[i] Einstein Static Universes: Parisi, Bruni, Maartens, Vandersloot (07)

Some Features of New Physics:

– Bounce occurs when $\rho = \rho_{\rm crit} \approx 0.82 \rho_{\rm Pl}$

– Inverse scale factor modifications play no role in singularity resolution. It occurs because of non-local effects from field strength operator.[a]

– Theory has correct classical limit. Difficulties regarding this for closed model,[b] overcome with new quantization.[c]

– Unlike the early works, QG effects occur at an invariant curvature scale.

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[a] Contrast with early claims: Bojowald (01-...), PS (05), ...

[b] Green, Unruh (05)

[c] Ashtekar, Pawlowski, PS, Vandersloot (06)

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Some Open Questions

- Is bounce restricted only to the states which are semi-classical at late times? What happens in the case of generic states?
- What happens to the fluctuations in general? Is the Universe on the other side quantum or classical?
- What is the significance of $\rho_{\rm crit}$?
- In what sense LQC and WDW converge to each other or diverge from each other?

Exactly Solvable LQC (sLQC) [a]

Based on a small and well motivated approximation. (Role of inverse triad modifications is negligible in singularity resolution). Full analytical control.

– Quantum Constraint in the conjugate (b) representation:

$$\Theta(b)\chi(b,\phi) \;=\; -12\pi G\,\frac{\sin(\lambda b)}{\lambda}\,\frac{\partial}{\partial b}\,\frac{\sin(\lambda b)}{\lambda}\,\frac{\partial}{\partial b}\,\chi(b,\phi) \;=\; -\,\partial_\phi^2\,\chi(b,\phi)$$

– Introduce $x := (12\pi G)^{-1/2}\,\ln(\tan(\lambda b/2))$

$\Rightarrow$

$$\partial_\phi^2\,\chi(x,\phi) = \partial_x^2\,\chi(x,\phi)$$

Wheeler-DeWitt:

$$\Theta(b)\chi(b,\phi) \;=\; -12\pi G\; b\frac{\partial}{\partial b}\, b\frac{\partial}{\partial b}\chi(b,\phi) \;=\; -\partial_\phi^2\chi(b,\phi)$$

– Introduce

$$y := (12\pi G)^{-1/2}\log\left(b/2b_o\right)$$

$\Rightarrow$

$$\partial_\phi^2\chi(\phi,y) = \partial_y^2\chi(\phi,y)$$

[a] Ashtekar, Corichi, PS (07)

Volume observable

Wheeler-DeWitt:

$$(\chi, \hat{V}|_\phi\, \chi)_{\rm phy} \;=\; 2\pi\gamma\ell_{\rm P}^2\, (\hat{\nu}\chi, \hat{\nu}\chi)_{\rm kin}$$
$$\;=\; V_o\, e^{\sqrt{12\pi G}\phi}\,.$$

– As $\phi \to -\infty$, $\langle \hat{V}|_\phi \rangle \to 0$. The backward evolution leads generically to the big bang singularity.

sLQC:

$$(\chi, \hat{V}|_\phi\, \chi)_{\rm phy} \;=\; V_+\, e^{\sqrt{12\pi G}\phi} + V_-\, e^{-\sqrt{12\pi G}\phi}$$

– As $\phi \to \pm\infty$, $\langle \hat{V}|_\phi \rangle \to \infty$. The Universe is infinitely large in asymptotic past and future.

– There exists a minimum value of $\langle V|_{(\phi=\phi_B)} \rangle$ which occurs at

$$\phi_B = (2\sqrt{12\pi G})^{-1}\, \ln(V_-/V_+)$$

Quantum Bounce is generic.

Results

- There exists an absolute upper bound on the energy density for any physical state in the Hilbert space: $\rho \leq \rho_{\rm sup} \approx 0.82\, \rho_{\rm Pl}$. The critical density $\rho_{\rm crit}$ obtained from numerical simulations turns out to be the supremum!
- For a finite 'time' interval, it is always possible to choose a value of λ such that the dynamics of WDW and sLQC agree to an arbitrary precision. However, a patient observer would see their sharp difference for any $\lambda > 0$ if he waits long enough. The global dynamics of WDW and sLQC is very distinct.
- Fluctuations:[a]
  - For a very large class of states universe retains all semi-classical features across the bounce:

$$\chi(x, \phi) = \int_0^\infty dk\ \tilde{F}(k)\, e^{-ik(\phi+x)} - \int_0^\infty dk\ \tilde{F}(k)\, e^{-ik(\phi-x)}$$

    For any real and arbitrary $\tilde{F}(k)$, fluctuations are symmetric.
  - For more general states, relative fluctuations in conjugate variables in post bounce phase puts very strong constraints on change in relative fluctuations in pre bounce phase. For a 1 Megaparsec universe: change $< 10^{-56}$. Universe retains semi-classicality across the bounce.[b]

[a] Corichi, PS (07)

[b] Also for an LQC inspired model: Bojowald (07)

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Summary and Open Issues

- Loop quantum cosmology provides a glimpse on the origin of the Universe in non-perturbative quantum gravity. Emerging picture from simple models:

  Big bang not the beginning, big crunch not the end.
  Two classical regions of spacetime joined by a
  quantum geometric bridge.

- Quantum gravity makes curvature non-local at Planck scale. This plays an important role to yield a non-singular evolution across the classical singularity. No need to introduce any exotic matter/ad-hoc assumptions/fine tuning.

- Bounce occurs for states in a dense subspace of the physical Hilbert space (not only for those which are semi-classical at late times).

- There exists an upper bound on the value of energy density at which the universe bounces. $\rho_{\rm sup} = \rho_{\rm crit} \to \infty$ as $G\hbar \to 0$. Bounce a pure quantum gravity effect.

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Summary and Open Issues

- The universe retains semi-classical properties across the bounce even for generic states.
- LQC and WDW approach GR at low curvatures. At large curvatures they depart significantly.
- What happens when we include anisotropies? Bounce picture unaffected.[a]
- Does the picture of the bounce survive when we include inhomogenities? Can perturbations be propagated across the bounce?
  Work on perturbations started.[b]
- What happens to the singularity resolution in more general spacetimes?[c]
- What is the analog of Raichaudhuri equation describing non-singular QG effects? Is there any Non-Singularity Theorem?
- What is the deeper principle which leads to singularity resolution? (Crucial to develop full QG)

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[a] Using Effective Hamiltonian: Chiou, Vandersloot (07)

[b] Preliminary Works: Bojowald, Hossain, Kagan, Nunes, Mulryne, PS, ... (06-...)

[c] In progress. Examples - Gowdy Models: Banerjee, Date (07)

Fluctuations[a]

For a very large class of states universe retains all semi-classical features across the bounce:

$$\chi(x,\phi) = \int_0^\infty dk\ \tilde{F}(k)\, e^{-ik(\phi+x)} - \int_0^\infty dk\ \tilde{F}(k)\, e^{-ik(\phi-x)}$$

For any real and arbitrary $\tilde{F}(k)$, fluctuations are symmetric.

Generic States: Consider a state in the present epoch (post big bang) describing a large classical universe at low curvature

$$\lim_{\phi\to\infty} (\Delta\hat{V}/\langle\hat{V}\rangle)^2 = (W_+/V_+^2) - 1 =: \delta_v \ll 1$$

Relative dispersion in curvature:

$$(\Delta \tan(\lambda b/2)/\langle\tan(\lambda b/2)\rangle) = \Delta x =: \delta_b \ll 1$$

$$D \;=\; (\Delta V/\langle\hat{V}\rangle)^2_{\phi\to-\infty} - (\Delta V/\langle\hat{V}\rangle)^2_{\phi\to\infty} < (1+\delta_v)\,(e^{8\sqrt{12\pi G}\ \delta_b} - 1)$$

For a universe which grows to the size of a MegaParsec, $D < 10^{-56}$. The change in relative fluctuations is negligible for a realistic universe! Universe retains semi-classicality across the bounce.[b]

[a] Corichi, PS (07)

[b] Also for an LQC inspired model: Bojowald (07)