Cosmic Microwave Background Radiation

Lecture 1 : Physics of CMB

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Plan of the Talk

- Standard Model of Cosmology
- Cosmic Microwave Background
- Boltzmann Equation
  - Phase Space density
  - Collision part
  - Collision-less part
  - Boltzmann equation for photons
- Einstein Equations
- Line of sight integration

Hot Big Bang Cosmology : Standard Model of Cosmology

- Large scale uniformity - Homogeneity and Isotropy - Hubble expansion
- Early Universe dense, hot and small - the Big Bang
- Gravitation only the relevant interaction at large scale - General Relativity

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu} \tag{1}$$

- Homogeneous and Isotropic space time - FRW metric:

$$ds^2 = c^2 dt^2 - a^2(t) \left[\frac{dr^2}{1 - kr^2} + d\theta^2 + \sin^2\theta d\phi^2 \right], \tag{2}$$

  only two parameters - scale factor $a(t)$ and spatial curvature k.

- Most of the energy of the Universe is in dark energy (70%) and dark matter (25%), very less in baryons or atoms (5%).
- Inflation

Friedman Equations

- For FRW metric, Einstein equation (1) can be written in terms of a pair of equations called Friedman equations

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G\rho}{3} \tag{3}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \tag{4}$$

- The rate of the expansion of the Universe is given by the Hubble parameter:

$$H(t) = \frac{1}{a(t)}\frac{da(t)}{dt} \tag{5}$$

- Energy density of any species is given by the density parameter $\Omega\rho/\rho_c$ where ρ_c is called the critical density and is defined as:

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G} \tag{6}$$

Distances

- Physical distance between objects in an expanding universe increases in proportion of $a(t)$:

$$\lambda(t) = \frac{a(t)}{a(t_0)}\lambda(t_0) \tag{7}$$

- In comoving coordinate system (which expands with the universe) distances between objects do not change with time due to expansion.
- The distance at which two objects in the Universe move away with each other with the speed of light is called the Hubble distance d_H:

$$d_H = \frac{c}{H} \tag{8}$$

- Comoving size of the Universe is given by η:

$$\eta = \int \frac{cdt}{a(t)} = \int_0^a \frac{da'}{a'} \frac{cda'}{a'H(a')} \tag{9}$$

Numbers

- Hubble parameter h is measured in 100 Km/sec/ Mpc[1]

$$H_0 = \frac{h}{0.98 \times 10^{10}\text{year}} \quad \text{where} \quad 0.5 < h < 1.0 \tag{10}$$

- Hubble distance :

$$d_H = \frac{c}{H_0} \approx 9449 \quad \text{Mpc/h} \tag{11}$$

- Critical density:

$$\begin{aligned} \rho_c = \frac{3H_0^2}{8\pi G} &= 1.88h^2 \times 10^{-29}\text{gm} \quad \text{cm}^{-3} \\ &= 2.775h^{-1} \times 10^{11} M_\odot/(h^{-1}\text{Mpc})^3 \end{aligned} \tag{12}$$

- Temperature:

$$T_{\rm CMB} = 2.725K \approx 2.35 \times 10^{-4} \ \ \text{eV} \tag{13}$$

[1] 1 Mpc $= 3.0856 \times 10^{24}$ cm

Observational Pillars

- Hubble expansion

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- Big Bang Nucleosynethis

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- Cosmic Microwave Background Radiation (CMB)

Cosmic Microwave Background

- The cosmic microwave background (CMB) was discovered by Wilson & Penzias [Penzias & Wilson (1965)] in 1965 and for this discovery they were awarded 1978 Nobel Prize in Physics.
- CMB was theoretically predicted in the context of synthesis (nuclear) of elements by Alpher and Herman [Alpher & Herman (1948)] and Gamow [Gamow (1948)] in late 1940s and again later rediscovered by Zelodovich, Dicke, Peebles [Dicke et al. (1965)].
- In early 1990s the COBE mission of NASA discovered [Smoot et al. (1992)] that the temperature of CMB is not the same along different direction, or there are anisotropies and for this John C. Mather and George F. Smoot were awarded 2006 Noble prize in physics.
- WMAP and Planck have further measured CMB anisotropies with great precision.

What we know about CMB ?

- CMB is a perfect blackbody radiation with temperature 2.725 degree Kelvin so its specific intensity is given by

$$I_\nu = \frac{2h^3}{c^2}\frac{1}{e^{h\nu/k_B T} - 1} \tag{14}$$

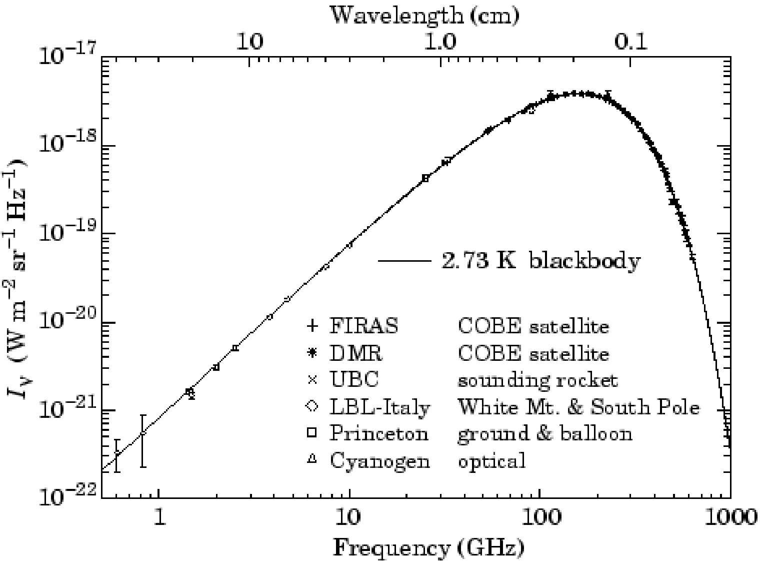
- Largest anisotropy 10^{-3} in the CMB sky is due to the motion of the solar system with respect to the rest frame of CMB (dipole) :

$$\frac{\Delta T}{T} = \frac{v}{c}\cos\theta \tag{15}$$

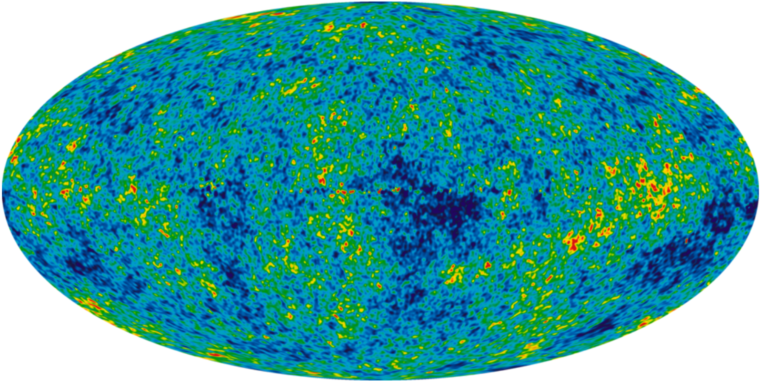
  for v=370 km/sec we get $\Delta T = 3.358 \times 10^{-3}$ Kelvin.

- Ignoring the dipole anisotropy, CMB anisotropies are of the order of 10^{-5}.
- CMB anisotropies are Gaussian.

CMB Black Body spectrum



CMB Anisotropies



- CMB anisotropies can be expressed in terms of multipole moments:

$$\frac{\Delta T(\hat{n})}{T} = \Theta(\hat{n}) = \sum_{l=0}^{l=\infty} \sum_{m=-l}^{m=l} a_{lm} Y_{lm}(\hat{n}) \tag{16}$$

with

$$a_{lm} = \int d\hat{n} Y^*_{lm}(\hat{n}) \Theta(\hat{n}) \tag{17}$$

and

$$\langle a^*_{lm} a_{l'm'} \rangle = \delta_{ll'} \delta_{mm'} C_l \text{ and } \quad < a_{lm} >= 0 \tag{18}$$

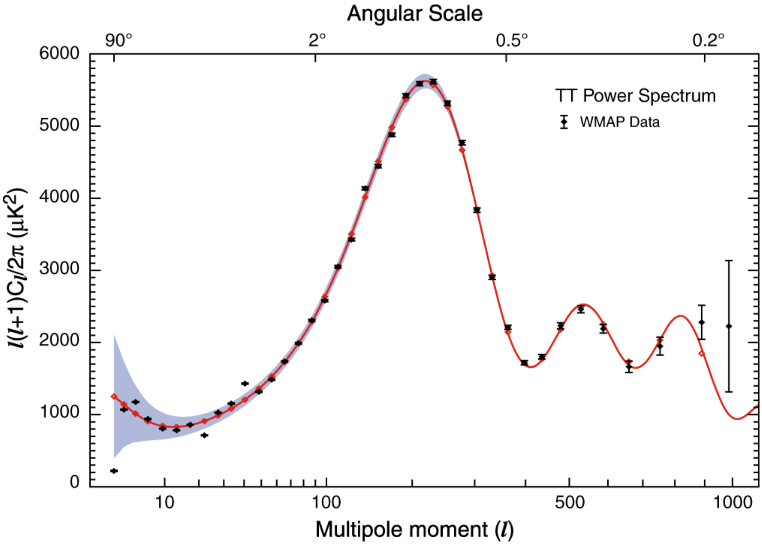
Where C_l is called angular power spectrum.

- C_l is directly related to two point angular correlation function:

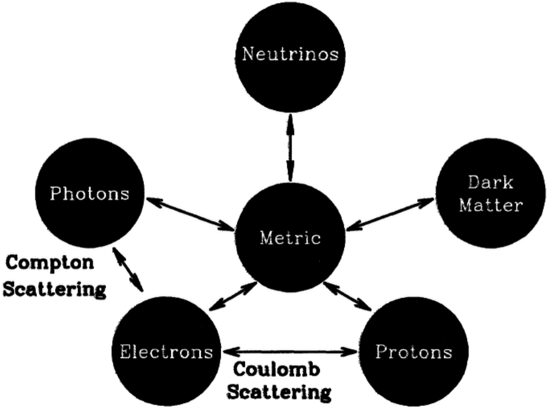
$$\left\langle \Theta(\hat{n}) \Theta(\hat{n}') \right\rangle = C(\theta) = \sum_l \frac{2l+1}{4\pi} C_l P_l(\cos\theta) \tag{19}$$

where $\cos\theta = \hat{n}.\hat{n}'$ and P_l are Legendre Polynomials.

CMB Anisotropies



Perturbations



CMB anisotropies are directly related to the fluctuations in matter and geometry of space time at the time of last scattering (primary) and after that (secondary).

- Phase space density $f(t, \vec{x}, \vec{p})$ also called distribution function, is defined in terms of the number of configurations within an infinitesimal phase volume d^3xd^3p around the point $(\vec{x}, \vec{p})$

$$dN = f(t, \vec{x}, \vec{p})d^3xd^3p \tag{20}$$

- In case of photons, which follow Bose-Einstein statistics, the distribution function is (thermal equilibrium) is just the function of energy:

$$f_{BE}(p) = \frac{1}{e^{pc/k_BT} - 1} \tag{21}$$

Number density and energy density

- The number density of photons at temperature T can be calculated as:

$$n_\gamma = 2\int \frac{d^3p}{(2\pi\hbar)^3} f(p) = 8\pi\left(\frac{k_B T}{hc}\right)^3 \int_0^\infty \frac{x^2 dx}{e^x - 1} \propto T^3 \tag{22}$$

  where prefactor 2 is for two polarizations.

- Energy density also can be calculated from the distribution function :

$$\rho_\gamma = 2\int \frac{d^3p}{(2\pi\hbar)^3}(pc) f(p) = \frac{8\pi^5 k_B^4}{15h^3c^3} T^4 = aT^4 = \frac{4\sigma}{c} T^4 \tag{23}$$

  with

$$\int_0^\infty \frac{x^2 dx}{e^x - 1} = 2\xi(3) = 2.404 \;\text{ and }\; \int_0^\infty \frac{x^3 dx}{e^x - 1} = 6\xi(4) = \frac{\pi^2}{15} \tag{24}$$

[Kolb & Turner (1990); Dodelson (2003); Weinberg (2008)]

Problem 1

Given that the CMB is a black body distribution with temperature 2.725 K show that:

- number density of CMB photons is around 440 /cc and
- energy density $\Omega_\gamma \approx 2.47 \times 10^{-4}/h^2$
- photon to baryon ratio is around 10^9.

Boltzmann Equation

- Boltzmann equation describes the evolution of phase space density, i.e., the distribution function $f(t, \vec{x}, \vec{p})$.

$$\frac{df(t, \vec{x}, \vec{p})}{dt} = C[f] \tag{25}$$

- The LHS is called the "collisionless" part and it describes the effect of gravity.
- The RHS is called the "collisional" part and it describes the change in the phase density due to interaction of of particles (absorption, emission and scattering).

In the present discussion we will try to solve the Boltzmann for three different cases:

1. Recombination

$$e^{-} + p \longleftrightarrow H + \gamma \tag{26}$$

2. Compton Scattering:

$$e^{-}(p) + \gamma(q) \longleftrightarrow e^{-}(p') + \gamma(q') \tag{27}$$

3. Metric Perturbations: Change in photon density due to metric perturbations.

Boltzmann Equation: Collisional Part

- Let us consider a reversible physical process in which two particles labelled as '1' and '2' react and produce two particles labeled as '3' and '4' which again produce '1' and '2'

$$1 + 2 \longleftrightarrow 3 + 4 \tag{28}$$

- If the number density and phase densities of particles '1','2','3' and '4' are n_1, n_2, n_3, n_4 and f_1, f_2, f_3, f_4 respectively then from the Boltzmann Equation the number density of particles '1' changes as :

$$\begin{aligned} a^{-3}\frac{d(a^3 n_1)}{dt} = & \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} \int \frac{d^3p_3}{(2\pi)^3 2E_3} \int \frac{d^3p_4}{(2\pi)^3 2E_4} \\ & \times \delta_D(E_1 + E_2 - E_3 - E_4)\delta_D(p_1 + p_2 - p_3 - p_4)\mathcal{M}^2 \\ & \times \{f_3 f_4[1 \pm f_1][1 \pm f2] - f_1 f_2[1 \pm f_3][1 \pm f4]\} \end{aligned} \tag{29}$$

- In LHS the factor of a^3 is because the volume increases as a^3 and factor a^{-3} is because density falls as a^{-3} when the Universe expands.

- The number density of any species can be computed as:

$$n_i = g_i \int \frac{d^3p}{(2\pi)^3} f(p) \tag{30}$$

where for the case when $e^{E/T} >> 1$ we can ignore the difference between bosons and fermions:

$$f(p) = \frac{1}{e^{E/T} - 1} \tag{31}$$

- We can compute the number density in two limits:

$$n_i(0) = \begin{cases} g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-m_i/T}, & \text{if} T << m_i \\ , g_i \frac{T^3}{\pi^2} & \text{if} T >> m_i \end{cases} \tag{32}$$

- Replacing the integral and the second line in equation (29) by $< \sigma v >$ where σ is the scattering and v is the velocity we get:

$$a^{-3}\frac{d(a^3 n_1)}{dt} = n_1^{(0)} n_2^{(0)} < \sigma v > \left[\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\right] \tag{33}$$

- LHS in Equation (33) is of the order $n_1/t \approx n_1 H$ and RHS of the order of $n_1 n_2 < \sigma v >$. If the reaction rate $n_2 < \sigma v >>> H$ then the RHS will be much larger and the particles can be in equilibrium.
- In Equation (33) the equality can be maintained if the individual terms in RHS cancel each other.

$$\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} = \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \tag{34}$$

This equation is called Nuclear Statistical Equilibrium (NSE) or Saha equation.

Recombination : Approximate solution of Boltzmann equation

- When the temperature of CMB falls below 1 eV electrons combine with protons and form neutral hydrogen:

$$e^- + p = H + \gamma \tag{35}$$

  and the number density of free electrons (which scatter with photons and so affect CMB) drops.
- We are interested in finding the change in the number density of free electrons during recombination and for which we can use the Boltzmann equation:

$$\frac{n_e n_p}{n_H} = \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}} \tag{36}$$

- It is more useful to express the number density of free electrons in terms of fraction:

$$X_e = \frac{n_e}{n_e + n_H} = \frac{n_p}{n_e + n_H} \tag{37}$$

- Now we can write an evolution equation for X_e from Equation (36):

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H}\left[\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\epsilon_0/T}\right] \tag{38}$$

  where $\epsilon_0 = m_e + m_p - m_H$ is the Binding energy of hydrogen atom.

- Expressing $n_e + n_H \approx n_b$ in terms of baryon-photon ratio η i.e., $n_b = \eta n_\gamma$ and using the fact that $n_\gamma \propto T^3$ equation (38) can be written as:

$$\frac{X_e^2}{1 - X_e} \approx 10^9 \left(\frac{m_e}{2\pi T}\right)^3 e^{-\epsilon_0/T} \approx 10^{15} \quad \text{when} \quad T = \epsilon_0 \tag{39}$$

- Since the RHS becomes very large so the equation is satisfied only when X_e is close to unity or all the atoms are ionized and for $X_e < 1$ we must solve the full Boltzmann equation.

Recombination : Exact solution of Boltzmann equation

The Full Boltzmann equation can be written as:

$$a^{-3}\frac{d(a^3 n_e)}{dt} = n_e^{(0)} n_p^{(0)} < \sigma v > \left[\frac{n_H}{n_H^{(0)}} - \frac{n_e^2}{n_e^{(0)} n_p^{(0)}}\right]$$

$$= n_b < \sigma v > \left\{(1 - X_e)\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\epsilon_0/T} - X_e^2 n_b\right\} \qquad (40)$$

which gives :

$$\frac{dX_e}{dt} = \left\{(1 - X_e)\beta - X_e^2 n_b \alpha^{(2)}\right\} \qquad (41)$$

with ionization rate β and the recombination rate $\alpha^{(2)}$ are given by:

$$\beta = \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\epsilon_0/T} \qquad (42)$$

and

$$\alpha^{(2)} =< \sigma v > \qquad (43)$$

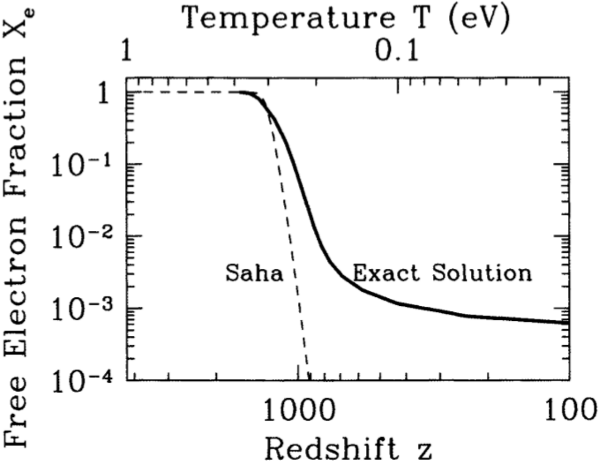


Figure 3.4. Free electron fraction as a function of redshift. Recombination takes place suddenly at $z \sim 1000$ corresponding to $T \sim 1/4$ eV. The Saha approximation, Eq. (3.37), holds in equilibrium and correctly identifies the redshift of recombination, but not the detailed evolution of X_e. Here $\Omega_b = 0.06, \Omega_m = 1, h = 0.5$.

- There is a superscript 2 on the recombination rate because recombination to the ground (n=1) is not useful since it leads to production of reionizing photon. and the only way for recombination to proceed is via capture to one of the excited states of hydrogen.
- The change in the number density of free electrons is important from the point of view of observational cosmology since recombination at $z^* \approx 1000$ is directly related to the decoupling of CMB photons.

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- Decoupling of CMB photons occurs roughly when the rate for photons to Compton scatter off electrons becomes smaller than the expansion rate.

$$n_e\sigma_T = X_e n_b \sigma_T = 7.477 \times 10^{-30}\ \mathrm{cm}^{-1} X_e \Omega_b h^2 a^{-3} \tag{44}$$

- Dividing the recombination rate by expansion rate (radiation dominated):

$$\frac{H}{H_0} = \Omega_m^{1/2} a^{-3/2} [1 + a/e_{eq}]^{1/2} \tag{45}$$

which gives:

$$\frac{n_e\sigma_T}{H} = 113 X_e \left(\frac{\Omega_b h^2}{0.02}\right)\left(\frac{.15}{\Omega_m h^2}\right)^{1/2}\left(\frac{1+z}{1000}\right)^{3/2}\left[1 + \frac{1+z}{3600}\frac{0.15}{\Omega_m h^2}\right]^{-1/2} \tag{46}$$

- Before recombination the main way by which photons and electrons were couples was Compton scattering:

$$e^{-}(\vec{q}) + \gamma(\vec{p}) \longleftrightarrow e^{-}(\vec{q}') + \gamma(\vec{p}') \tag{47}$$

- We are interested in finding the change in the phase space density for photons due to Compton scattering for the case $e^{E/T} >> 1$ (not differentiating bosons from fermions).

$$\begin{aligned} C[f(p)] = \frac{1}{p}\int \frac{d^3q}{2E_e(q)(2\pi)^3}\int \frac{d^3q'}{2E_e(q')(2\pi)^3}\int \frac{d^3p}{2E(p')(2\pi)^3}|\mathcal{M}|^2(2\pi)^4 \\ \times\delta^3(\vec{p}+\vec{q}-\vec{p}'-\vec{q}')\delta(E(p)+E_e(q)-E(p')-E_e(q)) \\ \times\{f_e(\vec{q}')f(\vec{p}')-f_e(\vec{q})f(\vec{p})\} \end{aligned} \tag{48}$$

- Note that here we are interested in a situation when photons are considered relativistic i.e., $E(p) = pc$ and electrons non-relativistic i.e., $E_e(q) = m_e c^2 + q^2/2m_e$.

Compton scattering

- Note that for first approximation we can ignore the direction dependence of Compton scattering and the amplitude for Compton scattering can be written as:

$$|\mathcal{M}|^2 = 8\pi\sigma_T m_e^2 \tag{49}$$

where σ_T is the Thomson scattering cross-section.

- The Dirac delta function for energy conservation (in the limit when very less energy is exchanged) can be approximated as:

$$\delta(E(p)+E_e(q)-E(p')-E_e(q)) \approx \delta(p-p') + \frac{(\vec{p}-\vec{p}').\vec{q}}{m_e}\frac{\partial\delta(p-p')}{\partial p'} \tag{50}$$

- We Taylor expand the distribution function and keep only the linear terms:

$$f(t,\vec{x},\vec{p}) = \left[e^{p/T(t)\{1+\Theta(t,\vec{x},\hat{p})\}} - 1\right]^{-1} \approx f^0(p) - p\frac{\partial f^0(p)}{\partial p}\Theta(t,\vec{x},\hat{p}) \tag{51}$$

where $\Theta(t,\vec{x},\hat{p})$ is also called the brightness function.

- With all these simplifications the collision term can be written as:

$$C[f(\vec{p})] = -p\frac{\partial f^0(p)}{\partial p} n_e \sigma_T \left[\Theta_0 - \Theta(\hat{p}) + \hat{p}.\vec{v}_b\right] \qquad (52)$$

where $\vec{v}_b = \vec{q}$ is the velocity vector for electron and Θ_0 is the temperature monopole[2] terms:

$$\Theta_0 = \frac{1}{4\pi}\int d\Omega' \Theta(t, \vec{x}, \hat{p}') \qquad (54)$$

- When electrons do not have bulk velocity Compton scattering try to drive temperature anisotropies towards the monopole i.e., $\Theta(\hat{p}) \longrightarrow \Theta_0$.

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[2] l^{th} monopole is defined as:

$$\Theta_l(\mu) = \frac{1}{(-i)^l}\int_{-1}^{1} \frac{d\mu}{2} P_l(\mu)\Theta(\mu) \qquad (53)$$

- In the derivation of the collision term due to Compton scattering we ignored (1) direction dependence of scattering and (2) Polarization.
- If these corrections are taken into account the collision term becomes:

$$C[f(\vec{p})] = -p\frac{\partial f^0(p)}{\partial p} n_e \sigma_T \left[\Theta_0 - \Theta(\hat{p}) + \hat{p}.\vec{v}_b - \frac{1}{2}P_2(\mu)\Pi\right] \tag{55}$$

where $\mu = \hat{p}.\hat{v}_b$, $P_2(\mu)$ is the Legendre polynomial and Π is defined as:

$$\Pi = \Theta_2 + \Theta_{P0} + \Theta_{P2} \tag{56}$$

where Θ_2, Θ_{P2} are the dipole components of the temperature and polarization fields respectively, and Θ_{P0} is the monopole part for polarization.

CMB Theory : Metric perturbations

- Geometric structure of a homogeneous and isotropic Universe is given by the Friedman-Robertson-Walker (FRW) metric and for spatially flat case this can be written as:

$$ds^2 = -c^2 dt^2 + a^2(t)\delta_{ij} dx^i dx^j \tag{57}$$

- There is a theorem called the decomposition theorem which says that perturbations to the metric can be divided up into three types: scalar, vector, and tensor and of these type evolves independently.
- Scalar perturbations (in conformal Newtonian Gauge) are represented by two functions $\Psi(\vec{x}, t)$ and which $\Phi(\vec{x}, t)$ which corresponds to perturbations in Newtonian potential and spatial curvature respectively.

$$ds^2 = -[1+2\Psi(\vec{x}, t)]c^2 dt^2 + a^2(t)\delta_{ij}[1+2\Phi(\vec{x}, t)] dx^i dx^j \tag{58}$$

- The LHS side of Boltzmann can be explicitly written as:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt} + \frac{\partial f}{\partial \hat{p}^i}\frac{d\hat{p}^i}{dt} \tag{59}$$

- In the linear order perturbation theory we can ignore the dependency of f on $\hat{p}^i$ so we get:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt} \tag{60}$$

- We must compute the velocity dx^i/dt and the force dp/dt in the perturbed metric given by Equation (58).
- Evolution equation i..e., Boltzmann equation for distribution function becomes evolution equation for temperature anisotropies in linear order perturbation:

$$f(t, \vec{x}, \vec{p}) \approx f^0(p) - p\frac{\partial f^0(p)}{\partial p}\Theta(t, \vec{x}, \hat{p}) \tag{61}$$

Four Momentum

- Before we compute the velocity and acceleration it is useful to find the components of the four momentum P^μ and corresponding temporal and spatial components.

$$P^\mu P^\mu = g_{00}(P^0)^2 + p^2 = -(1+2\Psi)(P^0)^2 + p^2 \tag{62}$$

or

$$P^0 = \frac{p}{\sqrt{1+2\Psi}} \approx p(1-\psi) \tag{63}$$

- For spatial part we can write:

$$P^i = C\hat{p}^i \tag{64}$$

where C is a constant which we can compute in the following way:

$$p^2 = P^i P_i = C^2 g_{ij}\hat{p}^i\hat{p}^j = C^2 a^2(1+2\Phi) \tag{65}$$

and so

$$C = \frac{p}{a\sqrt{1+2\Phi}} \tag{66}$$

and

$$P^i = \frac{p}{a}(1-\Phi)\hat{p}^i \tag{67}$$

Four Momentum

- Velocity can be written as:

$$\frac{dx^i}{dt} = \frac{dx^i}{d\lambda}\frac{d\lambda}{dt} = \frac{P^i}{P^0} = \frac{1}{a}(1 + \Psi - \Phi)\hat{p}^i \tag{68}$$

- For momentum we must use the Geodesic equation:

$$\frac{dP^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{69}$$

  For time-time component:

$$\frac{dP^0}{d\lambda} + \Gamma^0_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{70}$$

  where

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\frac{\partial g_{\nu\alpha}}{\partial x^\beta} + \frac{\partial g_{\nu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right) \tag{71}$$

- From equation (70) and metric given by equation (58) we can find out:

$$\frac{1}{p}\frac{dp}{dt} = -H - \frac{\partial\Phi}{\partial t} - \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i} \tag{72}$$

Boltzmann Equation for photons

- The collisionless part of the Boltzmann equation can be written as:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - p\frac{df}{dp}\left[H + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right] \tag{73}$$

- In order to connect fluctuations in the distribution function to the fluctuations in temperature we expand the distribution perturbatevely around the equilibrium distribution:

$$f(t,\vec{x},\vec{p}) \approx f^0(p) - p\frac{\partial f^0(p)}{\partial p}\Theta(t,\vec{x},\hat{p}) + \ldots \tag{74}$$

with

$$f^0 = \frac{1}{e^{p/T} - 1} \tag{75}$$

Zero order perturbations

- The collision term is zero :

$$\frac{df}{dt} = 0 = \frac{\partial f}{\partial t} - p\frac{df}{dp}H \tag{76}$$

- The distribution function $f \approx f_0$ and

$$\frac{\partial f^0}{\partial p} = -\frac{T}{p}\frac{\partial f^0}{\partial T} \tag{77}$$

and

$$\frac{\partial f^0}{\partial t} = -\frac{p}{T}\frac{dT}{dt}\frac{\partial f^0}{\partial p} \tag{78}$$

- The zeroth order equation becomes:

$$\left[-\frac{1}{T}\frac{dT}{dt} - \frac{1}{a}\frac{da}{dt}\right]\frac{\partial f^0}{\partial p} = 0 \tag{79}$$

which gives:

$$T(t) \propto \frac{1}{a(t)} \tag{80}$$

Linear order perturbations

- If we keep only linear order terms (in Θ, Ψ, Ψ) in the LHS and RHS (due to Compton scattering) in the Boltzmann equation for photons we get:

$$\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i} = n_e \sigma_T [\Theta_0 - \Theta + \hat{p}.\vec{v}_b - \frac{1}{2} P_2(\mu)\Pi] \tag{81}$$

  This equation is also called the Brightness equation [Kurki-Suonio (2010)]

- In terms of conformal time the full Boltzmann equation can be written as:

$$\dot{\Theta} + \hat{p}^i \frac{\partial \Theta}{\partial x^i} + \dot{\Phi} + \hat{p}^i \frac{\partial \Psi}{\partial x^i} = n_e \sigma_T a [\Theta_0 - \Theta + \hat{p}.\vec{v}_b - \frac{1}{2} P_2(\mu)\Pi] \tag{82}$$

- We can expand temperature anisotropies $\Theta(\eta, \vec{x}, \hat{p})$ in Fourier space:

$$\Theta(\eta, \vec{x}, \hat{p}) = \int \frac{d^k}{(2\pi^3)} e^{ik\hat{k}.\hat{p}} \tilde{\Theta}(\eta, \hat{k}, k) \tag{83}$$

and write the Boltzmann equation (82) as:

$$\dot{\tilde{\Theta}} + ik\mu\tilde{\Theta} + \dot{\tilde{\Phi}} + ik\mu\tilde{\Psi} = -\dot{\tau}[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\tilde{v}_b - \frac{1}{2}P_2(\mu)\Pi] \tag{84}$$

where the optical depth τ is defined as:

$$\tau(\eta) = \int_{\eta}^{\eta_0} a(\tau) d\eta' n_e \sigma_T \tag{85}$$

so $-n_e\sigma_T a = \dot{\tau}$ and the direction of propagation of photon is given by $\mu = \hat{k}.\hat{p}$.

- The Boltzmann equation for the density δ and velocity v of dark matter particles is given by:

$$\dot{\tilde{\delta}} + ik\tilde{v} = 3\dot{\tilde{\Phi}} \tag{86}$$

and

$$\dot{\tilde{v}} + \frac{\dot{a}}{a}\tilde{v} = -ik\tilde{\Psi} \tag{87}$$

- For baryons :

$$\dot{\tilde{\delta}}_b + ik\tilde{v}_b = 3\dot{\tilde{\Phi}} \tag{88}$$

and

$$\dot{\tilde{v}}_b + \frac{\dot{a}}{a}\tilde{v}_b = ik\tilde{\Psi} + \frac{\dot{\tau}}{R}[\tilde{v}_b + 3i\tilde{\Theta}_1] \tag{89}$$

where $R = 3\rho_b/\rho_\gamma$

- For neutrinos (massless)

$$\dot{\tilde{\mathcal{N}}} + ik\mu\tilde{\mathcal{N}} = -\dot{\tilde{\Phi}} - ik\mu\tilde{\Psi} \tag{90}$$

Einstein Equations

- We will consider the following perturbed metric:

$$ds^2 = -c^2 dt^2[1+2\Psi(\vec{x},t)]+a^2(t)\delta_{ij}[1+2\Phi(\vec{x},t)]dx^i dx^j \quad (91)$$

- We will consider the following two components of Einstein equations:

$$G^0_0 = \frac{8\pi G}{c^4} T^0_0 \quad (92)$$

and the trace-less part of the spatial component:

$$G^i_j \left(\hat{k}_i \hat{k}^j - \frac{1}{3}\delta^j_i k^2\right) = T^i_j \left(\hat{k}_i \hat{k}^j - \frac{1}{3}\delta^j_i k^2\right) \quad (93)$$

Energy Momentum tensor

- The temporal part of the Energy momentum tensor for photons is given by:

$$T^0_0 = -2\int \frac{d^3p}{(2\pi)^3} p \left[f^0 - p\frac{\partial f^0}{\partial p}\Theta \right] = -\rho_\gamma(1+4\Theta_0) \quad (94)$$

- The trace-less part of the spatial component is given by:

$$T^i_j \left(\hat{k}_i \hat{k}^j - \frac{1}{3}\delta^j_i k^2 \right) = \sum_i g_i \int \frac{d^3p}{(2\pi)^3} \frac{p^2\mu^2 - p^2/3}{E_i(p)} f_i(\vec{p}_i) \quad (95)$$

- The components of Einstein equation are:

$$k^2\tilde{\Phi} + 3\frac{\dot{a}}{a}\left(\dot{\tilde{\Phi}} - \tilde{\Psi}\frac{\dot{a}}{a} \right) = 4\pi G a^2 [\rho_{dm}\tilde{\delta}_{dm} + \rho_b\tilde{\delta}_b + 4\rho_\gamma\tilde{\Theta}_0 + 4\rho_\nu\tilde{\mathcal{N}}_0] \quad (96)$$

$$k^2(\tilde{\Phi} + \tilde{\Psi}) = -32\pi G a^2 (\rho_\gamma\tilde{\Theta}_2 + \rho_\nu\tilde{\mathcal{N}}_2) \quad (97)$$

Boltzmann Equations

- In order to solve the set of 9 first order differential (Boltzmann-Einstein) equations we need initial conditions.
- Since variables depend on each other so we do not need initial conditions for all.
- In fact when considering $\Psi = -\Phi$ we need just one initial condition i.e., for Φ.
- Inflation which explain large scale uniformity of the CMB sky also provides a mechanism to create perturbations in Φ.
- In the very early universe $k\eta << 1$ i.e., modes are outside horizon, these equations become quite simple since we can ignore terms which have k and higher power of k.

- Note that temperature anisotropies in Fourier space can be written as:

$$\Theta(\eta, \vec{x}, \hat{p}) = \int \frac{d^k}{(2\pi^3)} e^{ik\hat{k}.\hat{p}} \tilde{\Theta}(\eta, \hat{k}, k) \tag{98}$$

- Assuming that the perturbations are axisymmetric around k we can write:

$$\tilde{\Theta}(\eta, k, \mu) = \sum_l (2l+1)(-i)^l \tilde{\Theta}_l(\eta, k) P_l(\mu) \tag{99}$$

and its inverse:

$$\tilde{\Theta}_l(\eta, k) = \frac{1}{(-i)^l} \int_{-1}^{1} \frac{d\mu}{2} P_l(\mu) \tilde{\Theta}(\eta, k, \mu) \tag{100}$$

- We can solve for various multipoles $\tilde{\Theta}_l(\eta, k)$ of CMB anisotropies.

Tightly coupled limit of the Boltzmann equation

- Before decoupling photons and baryons were tightly couples with each other i.e., interaction rate was much larger than the expansion.
- In the tight coupling limit only multipoles which were significant are the monopole Θ_0 and dipole Θ_1 and photons behaved just like perfect fluid which is described by the density (monopole) and velocity (dipole).
- It turned out that the evolution equation for the monopole and dipole can be written as:

$$\dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi} \tag{101}$$

and

$$\dot{\Theta}_1 - \frac{k\Theta_0}{3} = \frac{k\Psi}{3} + \dot{\tau}\left[\Theta_1 - \frac{iv_b}{3}\right] \tag{102}$$

These equation can be obtained by multiplying the full Boltzmann equation (84) by $P_0(\mu)$ and $P_1(\mu)$ respectively and carrying out the integration over μ.

- Combining two first order DE for monopole and dipole into a second order DE and use eliminate the v_b by using the velocity equation we get the following equation:

$$\ddot{\Theta}_0 + \frac{\dot{a}}{a}\frac{R}{1+R}\dot{\Theta}_0 + k^2 c_s^2 \Theta_0 = -\frac{k^2\Psi}{3} - \frac{\dot{a}}{a}\frac{R}{1+R}\dot{\Phi} - \ddot{\Phi} = F(k,\eta) \tag{103}$$

where c_s is the sound speed and defined in the following way:

$$c_s = \sqrt{\frac{1}{3(1+R)}} \tag{104}$$

and it depends on the baryon density.

- The fluid oscillates both in space and time and the period of oscillations depend on the sound speed and so on the baryon density.

- We can solve the second order ordinary differential equation (103) for photons by Green function method i.e., firstly we can find the solution for the homogeneous equation and then use that to solve with the source term.
- Ignoring the damping term the solution for the homogeneous equation are as follows:

$$S_1(k,\eta) = \sin[kr_s(\eta)]; \quad \text{and} \quad S_1(k,\eta) = \cos[kr_s(\eta)] \qquad (105)$$

where r_s is the sound horizon and is given by:

$$r_s(\eta) \equiv \int_0^{\eta} d\eta' c_s(\eta') \qquad (106)$$

- The approximate solution we have written gives us enough information about the location of the peaks:

$$k_p = \frac{n\pi}{r_s} \quad \text{where} \quad n = 1, 2, 3, .. \tag{107}$$

- From the equations we have used for the monopole and dipole we can see that the monopole and dipole are out of phase:

$$\Theta_0(k,\eta) \approx \cos[kr_s(\eta)]; \quad \text{and} \quad \Theta_1(k,\eta) = \sin[kr_s(\eta)] \tag{108}$$

Integral solution of the Boltzmann equation

- The Boltzmann equation for photons can be written as:

$$\dot{\Theta} + (ik\mu - \dot{\tau})\Theta = -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 + \mu v_b - \frac{1}{2}P_2(\mu)\Pi\right] = S \tag{109}$$

where

$$S = e^{-ik\mu\eta+\tau}\frac{d}{d\eta}[\Theta e^{ik\mu\eta-\tau}] \tag{110}$$

- We can solve equation (110) in the following way:

$$\Theta(\eta_0) = \Theta(\eta_{init})e^{ik\mu(\eta_{init}-\eta_0)}e^{-\tau(\eta_{init})+\tau(\eta_0)} + \int_{\eta_{init}}^{\eta_0} d\eta S e^{ik\mu(\eta-\eta_0)-\tau(\eta)} \tag{111}$$

The first term is zero since $\tau(\eta_0) = 1$ and $\tau(\eta_{init})$ is very large so we have:

$$\Theta(k,\mu,\eta_0) = \int_0^{\eta_0} S(k,\mu,\eta)e^{ik\mu(\eta-\eta_0)-\tau(\eta)} \tag{112}$$

- Multiplying equation (112) by $P_l(\mu)$ both side and integrating over μ we get:

$$\Theta_l(k,\eta_0) = (-1)^l \int_0^{\eta_0} d\eta S(k,\eta) e^{-\tau(\eta)} j_l[k(\eta-\eta_0)] \quad (113)$$

where we have used :

$$\int_{-1}^{1} P_l(\mu) e^{ik\mu(\eta-\eta_0)} = \frac{1}{(-i)^l} j_l[k(\eta-\eta_0)] \quad (114)$$

where j_l is the spherical Bessel function.

- There are two terms in the equation (113) - the source term and the geometrical term. Ths geometrical term can be computed in advance irrespective to the model i., the source term which make the computation fast.

CMB Angular power spectrum

- CMB anisotropies are produced by the inhomogenities which were present at the time of recombination:

$$\Theta(\hat{n}) = \int dD \Theta(\mathbf{x}) \delta(D - D_*) \tag{115}$$

where D_* is the comoving distance of the last scattering surface.

- In Fourier space:

$$\Theta(\hat{n}) = \int \frac{d^3k}{(2\pi)^3} \Theta(\mathbf{k}) e^{i\mathbf{k}.D_* \hat{n}} \tag{116}$$

- We can expand the Fourier modes in terms of sphenrical harmonics:

$$e^{i\mathbf{k}.D_* \hat{n}} = 4\pi \sum_{l,m} i^l j_l(kD_*) Y^*_{lm}(\hat{k}) Y_{lm}(\hat{n}) \tag{117}$$

Angular Power spectrum

- Using :

$$\Theta(\hat{n}) = \sum_{l=0}^{l=\infty} \sum_{m=-l}^{m=l} a_{lm} Y_{lm}(\hat{n}) \tag{118}$$

we get

$$a_{lm} = \int \frac{d^3k}{(2\pi)^3} \Theta(\mathbf{k}) 4\pi i^l j_k(kD_*) Y_{lm}(\hat{k}) \tag{119}$$

- The angular power spectrum C_l can be computed as:

$$\langle a^*_{lm} a_{l'm'} \rangle = 4\pi \delta_{ll'} \delta_{mm'} \int d\ln k j_l^2(kD_*) \Delta_T^2(k) \tag{120}$$

where $\Delta_T(k) = k^3 P(k)/2\pi^2$ is the spatial power spectrum i.e., $P(k) =< |\Theta(\mathbf{k})|^2 >$

- For the case of slowly varying power spectrum i.e., scale invariant, we can take $\Delta_T^2(K)$ out of integration and get:

$$C_l \approx \frac{2\pi}{l(l+1)} \Delta_T^2(l/D_*) \;\; \text{or} \;\; \Delta_T^2(l/D_*) = \frac{l(l+1)}{2\pi} C_l \tag{121}$$

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