Einstein's $E = mc^2$

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Abstract

The iconic equation of physics $E = mc^2$ is suppose to be given by Albert Einstein. In the present article I will discuss that the equation was around before Einstein proposed his special theory of relativity in 1905, and was obtained using the notions of non-relativistic mechanics. Lev Okun [1, 2, 3, 4], a physicist belonging to Landau's school of physics has pointed out that this equation is in conflict with relativistic mechanics. He also reiterated the fact that the increase in the mass of a body with its velocity is a misconception, there is just one mass and that is the mass which can be identified with the Newtonian mass and it does not change with the velocity. In the present article I will summarize the ideas of Okun and present my own analysis to show that why $E = mc^2$ is inconsistent and why the concept of relativistic mass is unnecessary. In order to highlight the differences between relativistic and non-relativistic mechanics I will also present some simple calculations which should be easy to follow by those people also who do not have any background of general relativity. At the end of the article I will present the results of a survey which I carried out to find the presence of $E = mc^2$ and relativistic mass in a large number of text books.

1 Introduction

Einstein did not write $E = mc^2$ first time, the equation was around even twenty five years before Einstein proposed special theory of relativity in 1905, and was derived on the basis of Maxwell's theory of Electromagnetic radiation. It was also written down by Henri Poincare in 1900, five years before Einstein formulated the special theory relativity. The idea that mass of a body increases with its velocity was given by Hendrik Lorentz and J.J. Thomson also on the basis of the kinetic energy of a freely moving charged body. Thomson in 1881 computed the correction in the mass a body due to velocity to the second order. Lorentz in 1899 further developed the idea of the change in the mass of a body with its velocity. It is also claimed that Olinto De Pretto, an industrialist from Vicenza, published the equation $E = mc^2$ in a scientific magazine *Atte* in 1903[8]. Interestingly most of these ideas were based on non-relativistic formula $\mathbf{p} = m\,\mathbf{v}$. Einstein was not very consistent in using the mass m (relativistic mass) and m_0 (mass in the rest frame). The idea that mass can be assigned to energy helped Einstein to develop the general theory of relativity. It was very clear to Einstein in his 1905 paper "on the relativistic motion of charged bodies" that no single proportionality constant can be assigned to acceleration-force relation and he introduced the longitudinal $m_{||}$ and transverse mass $m_{\perp}$ in place of the relativistic mass. Pauli in 1921 in his book *The theory of relativity* rejected the idea of longitudinal and transverse mass and pushed forward the idea of relativistic mass. Landau and Lifshitz in their classic text book *Classical Field Theory* completely ignored the idea of relativistic mass and used just one mass i.e., invariant mass (mass in the rest frame). Einstein was not very comfortable with the idea of relativistic mass and in a Letter to Lincoln Barnett on 19 June 1948 he wrote [1, 2]:

> It is not good to introduce the concept of the mass $M = m/\sqrt{1 - v^2/c^2}$ of a moving body for which no clear definition can be given. It is better to introduce no other mass concept than the 'rest mass' m. Instead of introducing M it is better to mention the expression for the momentum and energy of a body in motion.

In the prefeace of his book [3] Okun writes about $E = mc^2$:

This famous equation and the concept of mass increasing with velocity indoctrinate teenagers through the popular science literature, and through college text-books. According to Einstein, ”common sense is a collection of prejudices acquired by age eighteen. ”It is very difficult to get rid of this ”common sense” later: ”better untaught than ill taught.” As a result one can find the term ”rest mass” even in serious professional physics journals.

2 What is wrong with $E = mc^2$?

In relativistic physics the energy E and mass m belong to two very different class of objects. Energy is a component of momentum four vector $p^\mu = (E/c, \mathbf{p})$ and so is a coordinate dependent physical quantity. However, the mass m is a relativistically invariant physical quantity and so its values is the same in all coordinate systems. Note that mass is an invariant physical quantity but not conserved, however, energy is a conserved quantity but not invariant.The conservation of mass into energy is really a conversion of rest energy into kinetic energy [9].

The four momentum is given by

$$p^\mu = \left(\frac{E}{c},\ \mathbf{p}\right) \tag{1}$$

and

$$p^\mu p_\mu = \frac{E^2}{c^2} - p^2 = m^2 c^2 \tag{2}$$

or

$$E^2 - p^2 c^2 = m^2 c^4 \tag{3}$$

from this equation we can see that m gives the length of the momentum four vector p_μ and there is no need to put any subscript on it to specify the coordinate frame because it is relativistically invariant physical quantity. In the coordinate frame in which the particle is in rest or its linear momentum is zero, Equation (3) can be written as

$$E_0^2 = m^2 c^4 \tag{4}$$

or

$$\boxed{E_0 = mc^2} \tag{5}$$

In the above equation E_0 is the energy of the particle in the coordinate system in which its momentum is zero and this is the equation which people should be using in place of $E = mc^2$. In fact this is the equation which is relativistically consistent with equation (3) [2]

2.1 A simple derivation of $E = mc^2$

In order to prove $E = mc^2$ let us consider a pulse of light emitted at the left end of a box (see Figure (2.1)) and absorbed at the right end. When the light pulse is emitted the box starts moving in the opposite direction (since the center of mass has to be stationary) and stops moving when the light pulse reaches at the right end. If the length, area of cross section, volume and mass of the box are L, A, V and M respectively, the time of flight and mass of the light pulse are Δt and m respectively, we can compute the total distance Δx by which the box moves in the following way

Considering the energy density of light pulse in the box ρ and total energy E the pressure of light pulse is

$$P = \rho c^2 = \frac{E}{V} c^2 = \frac{E}{Ac\Delta t} c^2 \tag{6}$$

From the above equation we can compute the force acting on the box due to the reaction of the light pulse

$$F = -PA = \frac{E}{Ac\Delta t} c^2 A \tag{7}$$

and

$$v = a\Delta t = \frac{F}{M}\Delta t = -\frac{E}{Mc} \tag{8}$$

The distance travelled by the box

$$\Delta x = v\Delta t = -\frac{E}{Mc} \times \frac{L}{c} = \frac{EL}{Mc^2} \tag{9}$$

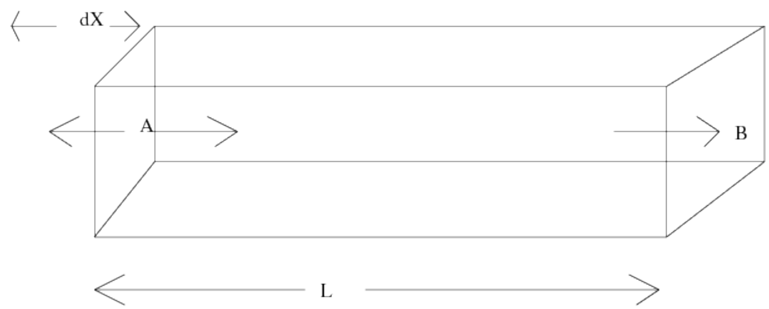

Figure 1: A photon is emitted at the end 'A' of the box and gets absorbed at the end 'B'. When the photon is emitted the box moves backward by an amount dx.

now since the center of mass does not move therefore

$$mL + M\Delta x = 0 \tag{10}$$

substituting the value of Δx we get

$$m = -\frac{M}{L}\Delta x = \frac{E}{c^2} \tag{11}$$

or

$$\boxed{E = mc^2} \tag{12}$$

This shows that a light pulse with energy E carried an equivalent mass m. The above derivation is based on non-relativistic consideration so obviously is not valid in relativistic domain. In fact Newton's third law i.e., the law of action-reaction no longer holds in relativistic physics [9].

2.2 Relativistic mass

One can derive the expression for the relativistic mass on the basis of $E = mc^2$ in the following way: From $E = mc^2$ we can write

$$c^2 dm = dE = Fds = Fvdt = vd(mv) = mvdv + v^2 dm \tag{13}$$

or

$$\int_{m_0}^{m} \frac{dm}{m} = \int_0^v \frac{vdv}{c^2 - v^2} \tag{14}$$

which gives

$$\boxed{m = \frac{m_0}{\sqrt{1 - v^2/c^2}}} \tag{15}$$

In the above equation m is called the relativistic mass and some of the text books (in particular books in particle physics and quantum field theory) has abandoned the use of it in favor of relativistic energy [9] which is given as

$$\boxed{E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}} \tag{16}$$

About the relativistic mass Adler [5] writes that it is the physical quantity which when multiplitied with velocity gives the relativistic momentum and it is not an inertia in a classical sense. Some of arguments against the use of relativistic mass are give in [6].

3 Relativistic Kinematics

Why massive objects cannot move with the speed of light ? The most common answer to this question is that the mass of objects increases rapidly when their speed approaches towards the speed of light. This simple answer is based on the assumption that acceleration and force are proportional and parallel to each other, which is not the case in relativistic mechanics. In relativistic mechanics $F = ma$ no longer remain valid.

The relativistic momentum is defined as

$$\mathbf{p} = m\gamma\ \mathbf{v} \text{ where } \ \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{17}$$

In relativistic case the Newton's second law (which remains valid) can be written as

$$\mathbf{F} = \frac{d\,\mathbf{p}}{dt} = \frac{d}{dt}\,(m\,\mathbf{v}\gamma) \tag{18}$$

or

$$\mathbf{F} = \frac{d}{dt}\,(m\,\mathbf{v}\gamma) = m\left(\frac{d\gamma}{dt}\,\mathbf{v} + \gamma\frac{d\,\mathbf{v}}{dt}\right) \tag{19}$$

substituting

$$\frac{d\gamma}{dt} = v\frac{dv}{dt}\frac{\gamma^3}{c^2} = \frac{\gamma^3}{c^2}(\,\mathbf{v}.\,\mathbf{a}) \tag{20}$$

so

$$\mathbf{F} = \frac{\gamma^3 m}{c^2}(\,\mathbf{v}.\,\mathbf{a})\,\mathbf{v} + \gamma m\,\mathbf{a} \tag{21}$$

$$\mathbf{F} = \frac{m}{\sqrt{1 - v^2/c^2}}\left[\mathbf{a} + \frac{(\,\mathbf{v}.\,\mathbf{a})\,\mathbf{v}}{c^2 - v^2}\right] \tag{22}$$

This equation shows that in relativistic case there is no $F = ma$ correspondence of $F = dp/dt$. In place of that if we write

$$\mathbf{F} = \frac{m}{\sqrt{1 - v^2/c^2}}\left[\mathbf{a}_{||} + \ \mathbf{a}_{\perp}\frac{1}{(1 - v^2/c^2)}\right] \tag{23}$$

or

$$F_{||} = m_{||}a_{||} \tag{24}$$

and

$$F_{\perp} = m_{\perp}a_{\perp} \tag{25}$$

Where the longitudinal mass $m_{||}$ and the transverse mass $m_{\perp}$ are given by

$$m_{\perp} = \gamma^3 m = \frac{m}{\left(\sqrt{1 - v^2/c^2}\right)^3} \tag{26}$$

and

$$m_{||} = \gamma m = \frac{m}{\sqrt{1 - v^2/c^2}} \tag{27}$$

These are the longitudinal mass $m_{||}$ and the transverse mass $m_{\perp}$ which were given by Albert Einstein and he also pointed out that they are not unique [7].

From equation (22) we can write

$$\mathbf{F} = \gamma\left[\mathbf{f} + \frac{(\,\mathbf{v}.\,\mathbf{f})\,\mathbf{v}}{c^2 - v^2}\right] \tag{28}$$

where $\mathbf{f} = m\,\mathbf{a}$

From the above equation we can see that apart from a component parallel to Newtonian force, relativistic force has a component parallel to velocity also. In fact the trajectory of particle being parabolic in the case of constant force, it is hyperbolic in a relativistic case as we will see below. In order to explain the impossibility of objects moving with the speed of light one should use the above formula in place of $f = ma$.

3.1 Relativistic motion of a particle under constant force

We know that a non-relativistic particle moving under constant force follows a parabolic path which is given by

$$\boxed{x(t) = \frac{1}{2}\frac{F}{m}t^2} \tag{29}$$

Now for a relativistic particle under constant force

$$\frac{dp}{dt} = F = \text{ constant} \tag{30}$$

so

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}} = Ft \tag{31}$$

or

$$v(t) = \frac{(Ft/m)}{\sqrt{1 + (Ft/mc)^2}} \tag{32}$$

or

$$x(t) = \int_0^t v(t')dt' = \int_0^t \frac{(Ft'/m)}{\sqrt{1 + (Ft'/mc)^2}}dt' \tag{33}$$

this can be easily integrated by substitution

$$x(t) = \frac{mc^2}{F}\left[\sqrt{1 + \left(\frac{Ft}{mc}\right)^2} - 1\right] \tag{34}$$

This is an equation of hyperbola. The relativistic motion of a particle under constant force is hyperbolic in place of parabolic. For $Ft << mc$ we equation (34) reduces into equation (29).

From the above discussion it is clear that if we write $F = ma$ for the longitudinal and transverse motions, we need to use longitudinal and transverse masses and which are different. In fact this is the reason that horizontally moving photon is twice heavy in comparision to a vertically moving photon in the field of massive body like the sun or earth. This is the extra factor of two which gives the correct angle of deflection for a photon passing through the field of a massive body.

The presence of $E = mc^2$ and relativistic mass $m = m_0\gamma$ in some text books [1]

4 Summary and conclusion

In the present article I reviewed the history of the famous formula $E = mc^2$ and discussed that it was not given by Albert Einstein as is popularly known. I showed that energy and mass belong to two very different class of objects in relativistic mechanics and so equating them is misleading. I have also elaborated on the issue of relativistic mass and discussed why the use of it is unnecessary. I have also showed that most of the misconceptions like $E = mc^2$ and $m = m_0\gamma$ arise when non-relativistic notions are used in relativistic domain like $p = mv$. I showed that in relativistic regime we cannot write down a single proportionality constant between force and acceleration. In the last section I presented the results of a survey I carried out the presence of $E = mc^2$ and $m = m_0\gamma$ in common text books. he main conclusions of the article are as follows:

- The mass m and energy E belongs to two different class of objects in relativistic mechanics and so equating them is misleading. Mass is an invariant and non-conserved physical quantity. However, energy is a conserved physical quantity but is not invariant.
- Since mass is defined in terms of the mod of Four momentum therefor it is relativistically invariant and specifying any coordinate system for it is misleading.
- In relativistic mechanics acceleration is not parallel to Newtonian force and so we cannot define a single proportionality constant. However, if we wish we can define longitudinal and transverse mass but they will not be unique.

[1] Please note that the data was taken in a mechanically way i.e., I scanned the text books for the above formula. For example, any book which does not write these formula explicitly but does mention these using text will not be considered having these formula. Therefore the data should not be interpreted that the author of the particular book believes in these formula or not. However, the presence and absence of these formula can be interpreted as the importance of these formula given by the author

S.NO	Author	Book	$m = m_0\gamma$?	$E = mc^2$?
1	Max Born (1920)	Einstein's theory of relativity	Yes	Yes
2	W. Pauli (1958 Reprint)	Theory of relativity	Yes	Yes
3	Dominico Giulini (2005)	Special Relativity	Yes	Yes
4	Lewis Ryder (2009)	General Relativity	No	No
5	Bernard Schutz (2009)	General Relativity	No	No
6	Albert Einstein (1916, Penguin 2006)	Relativity	No	No
7	Schwartz and Schwartz (2004)	Special Relativity	No	Yes
8	Sean Carroll (2003)	Spacetime and Geometry	No	Yes
9	HOBSON and EFSTATHIOU (2006)	General Relativity	Yes	Yes
10	Steven Weinberg (1972)	Gravitational and Cosmology	No	No
11	N. M.J. Woodhouse (2007)	General Relativity	No	No
12	Misner, Throne and Wheeler (1973)	Gravitation	No	No
13	Hawking and Ellis (19723)	LSS of spacetime	No	No
14	Maclcolm Ludvingsen (2004)	General Relativity	No	No
15	Francis E. Law (2004)	Classical Field Theory	No	No
16	McGlinn (2003)	Introduction to relativity	Yes	Yes
17	Harvey R. Brown (2005)	Physical Relativity	No	No
18	Bartrand Russel	ABC of relativity	Yes	No
19	Arthur Beiser	Concepts of Modern Physics	Yes	Yes
20	Ehlers and Lammerzahl (2006)	Special Relativity	No	No
21	Grown and Harvik (2007)	Einstein's General Theory of Rel.	Yes	Yes
22	Khriplovich (2005)	General Relativity	No	No
23	Bertel Laurnet (1994)	Introduction to spacetime	No	No
24	Cardone and Mignani (2004)	Energy and Geometry	Yes	Yes
25	D.-E. Liebscher (2005)	The Geometry of Time	Yes	Yes
26	Libber and Libber (1966)	Einstein's theory of relativity	No	No
27	Moses Fayngold (2002)	STR and motions faster than light	Yes	Yes
28	Becchi and D'Elia (2007)	Intro to basic concepts in Mod. Phys.	No	No
29	Ta. Pai Cheng (2005)	Relativity, Gravity and Cosmology	No	No
30	Andrew Liddle (2003)	An Intro. to modern cosmology	Yes	Yes
31	Forshaw and Smith (2009)	Dynamics and Relativity	No	No
32	Feynman (1965)	Lecture in physics	Yes	Yes
33	Griffiths (1980)	Electrodynamics	No	No
34	Jackson	Electrodynamics	Yes	Yes
35	Goldstein	Classical mechanics	No	No
36	Landau and Lifshitz	Classical Theory of Fields	No	No
37	David Bohn (1965)	The special theory of relativity	Yes	No [2]
38	James B. Hartle (2003)	Gravity	No	No
39	Wolfgang Rindler (1982)	Intro. to spec. relativity	Yes	Yes
40	Wolfgang Rindler (2006)	Relativity	Yes	Yes

References

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