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Cosmology

Jayanti Prasad

Chapter 1

Physical units & dimensions

1.1 Physical constants

S. No	Quantity	Symbol & relation	Value
1	Speed of light	c	$3.00 \times 10^{8}\ mt\ sec^{-1}$
2	Gravitational Constant	G	$6.67 \times 10^{-11}\ mt^{3}\ kg^{-1}\ sec^{-2}$
3	Planck's Constant	h	$6.62 \times 10^{-34}\ J\ sec$
		h	$4.13 \times 10^{-15}\ eV\ sec$
		$\hbar$	$1.05 \times 10^{-34}\ J\ sec$
		$\hbar$	$6.58 \times 10^{-16}\ eV\ sec$
4	Boltzmann Constant	k_B	$1.38 \times 10^{-23}\ J\ K^{-1}$
		k_B	$8.61 \times 10^{-5}\ eV\ K^{-1}$
5	Elementary charge	e	$1.60 \times 10^{-19}\ C$
6	Mass of proton	m_p	$1.67 \times 10^{-27}\ kg$
		m_p	$938.27\ Mev$
7	Mass of electron	m_e	$9.10 \times 10^{-31}\ kg$
		m_e	$0.51\ Mev$
8	Avagadro number	N_A	6.022×10^{23}
9	Stephan-Boltzmann cons	$\sigma = \pi^2 k_B^4/60\hbar^3 c^2$	$5.67 \times 10^{-8}\ Wt\ mt^{-2}\ K^{-4}$
10	Fine structure constant	$\alpha = (1/4\pi\epsilon_0)(e^2/\hbar c)$	7.29×10^{-3}
11	Rydberg constant	$R_\infty hc$	$13.60\ eV$
12	Bohr radius	$a_0 = 4\pi\epsilon_0\hbar^2/m_e e^2$	$0.53 \times 10^{-10}\ mt$
13	Classical electron radius	$r_e = \alpha^2 a_0$	$2.81 \times 10^{-15}\ mt$
14	Compton wavelength	$\lambda_c = h/m_e c$	$2.42 \times 10^{-12}\ mt$
15	Planck length	$l_P = \sqrt{\hbar G/c^3}$	$1.61 \times 10^{-35}\ mt$
16	Planck mass	$m_p = \sqrt{\hbar c/G}$	$2.17 \times 10^{-8}\ kg$
17	Planck time	$t_p = \sqrt{\hbar G/c^5}$	$5.39 \times 10^{-44}\ sec$
18	Thompson cross section	$\sigma_e = (8\pi/3) r_e^2$	$0.66 \times 10^{-28}\ mt^{2}$

1.2 Astrophysical & Cosmological constants

S. No	Quantity	Symbol & relation	Value
1	Astronomical unit	AU	$1.49 \times 10^{11}\ mt$
2	Light year	lyr	$9.46 \times 10^{15}\ mt$
3	Parsec	psc	$3.08 \times 10^{16}\ mt$
4	Mass of Sun	$M_\odot$	$1.98 \times 10^{31}\ kg$
5	Radius of Sun	$R_\odot$	$6.95 \times 10^{8}\ kg$
6	Luminosity of Sun	$L_\odot$	$3.85 \times 10^{33}\ erg\ sec^{-1}$
7	Mass of Earth	$M_\oplus$	$5.97 \times 10^{24}\ kg$
8	Radius of Earth	$R_\oplus$	$6.37 \times 10^{6}\ mt$
9	Hubble constant	H_0	$\approx (h/3000)\ Mpc^{-1}$
10	Hubble time	t_H	$9.78\ h^{-1}\ Gyrs$
11	Hubble size	cH_0^{-1}	$2998\ h^{-1}\ Mpc$
12	Critical density	$\rho_c = (3H^2/8\pi G)$	$2.775\ h^{-1}\ \times 10^{11}\ M_\odot/(h^{-1}\ Mpc^3)$

1.3 Gaussian Units

The Columb law for the force acting between two charges q and q' separated by a distance r is

$$F = k_1 \frac{qq'}{r^2} \quad \text{here} \ \ k_1 \text{ is a constant} \tag{1.1}$$

We can also define the electric field intensity E for a charge q at a distance r from the charge by the following expression:

$$E = k_1 \frac{q}{r^2} \tag{1.2}$$

Amperes law of the force per unit length acting between two infinite long, parallel thin charged conductors, carrying currents I, I', and separated by a distance d is given by the following expression:

$$\frac{dF}{dl} = 2k_2 \frac{II'}{d} \quad k_2 = \ \text{constant} \tag{1.3}$$

Since current $I = \frac{dq}{dt}$ so comparing the dimensions of F from equation (1.1) and equation (1.3) we find

$$\text{Unit} \left[\frac{k_1}{k_2} \right] = [L^2 T^{-2}] \tag{1.4}$$

Now one can guess that in vacuum

$$\frac{k_1}{k_2} = c^2 \tag{1.5}$$

In order to fix the units of k_1 and k_2 we need to use the following two more concepts:

a). When electric current I flows in a conductor then there induced a magnetic field with intensity B at a distance d which is given by:

$$B = 2k_2 \alpha \frac{I}{d} \quad \alpha = \ \ \text{constant} \tag{1.6}$$

b). Time varying magnetic field produces electric field and the intensity E of this field can be computed by the following expression:

$$\nabla \times E = -k_3 \frac{dB}{dt} \tag{1.7}$$

Now if we put the value of E and B in equation (1.7) and compare the units of LHS and RHS, and use the fact that anything which has dimensions $[LT^{-1}]$ can be replaced by c then we find the following expression:

$$\frac{k_1}{k_2 k_3 \alpha} = c^2 \tag{1.8}$$

From equation (1.5) and equation (1.8)

$$k_3 = \frac{1}{\alpha} \tag{1.9}$$

From the above equations it clear that out of k_1, k_2 and α we can normalize anyone of them unity. On the basis of the following two choices we have two systems of unit.

- **SI Units**

  We chose $\alpha = k_3 = 1$ and use

$$k_1 = \frac{1}{4\pi\epsilon_0} = 10^{-7}c^2[ML^3T^{-4}I^{-2}]$$

  and

$$k_2 = \frac{\mu_0}{4\pi} = 10^{-7}[MLT^{-2}I^{-2}]$$

- **Gaussian Units**

  We chose $k_1 = 1$ and this gives

$$k_2 = \frac{1}{c^2}, \quad \alpha = c\ [LT^{-1}] \ \text{ and } \ k_3 = \frac{1}{c}\ [L^{-1}T]$$

Law	SI	Gaussian
Coulomb's Law	$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$	$F = \frac{qq}{r^2}$
Ampere's Law	$B = \frac{\mu_0}{4\pi}\frac{Idl\sin\theta}{r^2}$	$B = \frac{1}{c}\frac{Idl\sin\theta}{r^2}$
Electric Displacement	$\mathbf{D} = \epsilon_0\ \mathbf{E}$	$\mathbf{D} = \frac{\mathbf{E}}{4\pi}$
Magnetic Intensity	$\mathbf{H} = \frac{\mathbf{B}}{\mu_0}$	$\mathbf{H} = \frac{c^2\ \mathbf{B}}{4\pi}$
Coulomb's Law	$\nabla.\mathbf{E} = \frac{\rho}{\epsilon_0}$	$\nabla.\mathbf{E} = \rho$
No monopole	$\nabla.\mathbf{B} = 0$	$\nabla.\mathbf{B} = 0$
Faraday's Law	$\nabla\times\mathbf{E} + \frac{\partial\ \mathbf{B}}{\partial t} = 0$	$\nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial\ \mathbf{B}}{\partial t} = 0$
Ampere's Law	$\nabla\times\mathbf{B} + \mu_0\left(\mathbf{J} + \epsilon_0\frac{\partial\ \mathbf{E}}{\partial t}\right) = 0$	$\nabla\times\mathbf{B} + \frac{4\pi}{c^2}\left(\mathbf{J} + \frac{1}{4\pi}\frac{\partial E}{\partial t}\right) = 0$

Chapter 2

Special Relativity

2.1 Introduction

Newtonian mechanics is based on the idea that all physical interactions (effects) like gravitation do not take time to travel from one place to another i.e., they are instantaneous. Einstein's special theory of relativity rejects this idea and postulates that nothing can travel faster than the speed of light in vacuum, which is a constant. This forces us to change our many physical notions particularly the notion of simultainty: Two physical events which are simultaneous in one reference frame no longer remain the same in another reference frame which is moving with a constant velocity with respect to the first, when we use special theory of relativity. There are two main postulates of special theory of relativity which are as follows:

- The motion of an object is identical in all the reference frames (inertial reference frames) which are moving with respect to each other with a constant velocity. This is called the **principle of relativity**. In the most general form, this postulate is stated as that the physical laws have the same form in all inertial reference frames, which are defined in the way that in these an unaccelerated particle moves along a straight line. In practical, all non-accelerated inertial frames are inertial.
- The speed of light in vacuum is a constant and nothing can move faster than that.

We need four coordinates to define any physical event: three spatial coordinates (x, y, z) and one time t. In Newtonian physics time is absolute in the sense that if two events take place at a time interval of dt in one reference frames (let us say S) then they will take place at the same time interval in another reference frame (let us call S') also which is moving with a constant velocity v along the x-axis with respect to the first one. Note that this is the consequence of the fact that Newton's laws of motion are invariant under the Galilean transformation which is defined in the following way:

$$x' = x - vt; \quad y' = y; \quad z' = z; \quad t' = t \tag{2.1}$$

One of the main motivations behind the discovery of special theory of relativity was that the Maxwell's equations of electromagnetism were found to be invariant under a different class of coordinate transformation (Lorentz transformation) in place of Galilean transformation. Note that under Galilean transformation, if two objects A and B are moving with velocities v_{OA} and v_{OB} with respect to an observer O then the objects A will move with velocity $v_{AB} = v_{OA} - v_{OB}$ with relative to B, however, under Lorentz transformation, we get a very complex velocity addition law which does not allow any velocity to cross the upper bound i.e., the velocity of light.

2.1.1 Four dimensional Minkowski space

In special theory of relativity space and time are taken as the same footing and so in place of using a three dimensional vector $x^i(i = 1, 2, 3)$ and time t to specify an event , we use a four dimensional vector $x^\mu(\mu = 0, 1, 2, 3)$ to specify the event. Note that all physical quantities in special theory of relativity are either taken as components of some four vectors or their combinations. The space consisted by three spatial dimensions and one temporal dimension is called the Minkowski space. The components of a four vector in the four dimenionsl Minkowski space depend on the choice of reference frame (coordinate system), as in three dimenionsl space, so it is important to know how to transform these components from one inertial reference frame to another.

In three dimensional space when we go from one ference frame to another (from S to S') then the mode of a vector $\vec{dl}$ (or dl^i where i=1,2,3) which is the infinitesimal distance between two points (x_1, x_2, x_3) and $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$ remain the same.

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 = \sum_{i=1,2,3} dx_i^2 = \eta_{ij} dx^i dx^j \tag{2.2}$$

$$dl'^2 = dx'^2_1 + dx'^2_2 + dx'^2_3 = \sum_{i=1,2,3} dx'^2_i = \eta_{ij} dx'^i dx'^j \tag{2.3}$$

and

$$dl^2 = \eta_{ij} dx^i dx^j = dl'^2 = \eta_{ij} dx'^i dx'^j \tag{2.4}$$

Note that here we have used the convention that repeated indices are summed and η_{ij} is a three dimensional matrix (called the metric) and is is defined in the following way:

$$\eta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.5}$$

In the four dimensional Minkowski space the distance between two space-time points (x_1, x_2, x_3, t) and $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3, t + dt)$ is defined as

$$ds^2 = c^2 dt^2 - dx_1^2 - dx_2^2 - dx_3^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{2.6}$$

note that here

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.7}$$

Here ds^2 is the four dimensional distance (also called the line element) which remains invariant when we go from one inertial frame to another and $\eta_{\mu\nu} = diag(1, -1, -1, -1)$ is called the Minkowski metric. On the basis of either $ds^2 > 0, < 0$ or $= 0$, the distance between two events is said to be time-like, space-like and light like respectively.

2.1.2 Lorentz transformations

So far we have not discussed that how coordinates of a point in Minkowski space are transformed when we go from an inertial frame S to another inertial frame S' which moves along the axis x with velocity v with respect to S. In this section we will find these transformations on the basis of the following considerations.

- These transformations must be linear so that the homogeneity in space and time is respected i.e., there is no special place or time in the universe.
- They must keep the four dimensional line element dl^2 invariant.

On the basis of these considerations we guess the following coordinate transformations:

$$t' = \beta x + \gamma t; \quad x' = \alpha(x - vt); \quad y' = y; \quad z' = z \tag{2.8}$$

Now

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{2.9}$$

and

$$ds'^2 = c^2d't^2 - dx'^2 - dy'^2 - d'z^2 \tag{2.10}$$

substituting the values of (x', y', z', t') we get

$$ds'^2 = c^2(\beta dx + \gamma dt)^2 - \alpha^2(dx - vdt)^2 - dy^2 - dz^2 \tag{2.11}$$

Now using $ds^2 = ds'^2$ a comparing the cofficents of dx^2, dt^2 and $dxdt$ from both sides:

$$\alpha^2 - c^2\beta^2 = 1 \tag{2.12}$$

$$\gamma^2 - \frac{v^2}{c^2}\alpha^2 = 1 \tag{2.13}$$

$$\beta\gamma c^2 + \alpha^2 v = 0 \ \text{ or } \ \gamma^2 = \frac{\alpha^2 v^2}{\beta^2 c^4} \tag{2.14}$$

Frome equation (2.13) and (2.14)

$$\alpha^4 v^2 - \alpha^2 v^2 c^2 \beta^2 = \beta^2 c^4$$

Now substituting the value of $c^2\beta^2$ from (2.12) we get:

$$\alpha = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.15}$$

Substituting the value of α in (2.13) we find:

$$\gamma = \alpha \tag{2.16}$$

Now from the above equations and (2.14)

$$\beta = -\gamma \frac{v}{c^2} \tag{2.17}$$

$$t' = \gamma(t - vx/c^2); \quad x' = \gamma(x - vt); \quad y' = y; \quad z' = z; \text{ where } \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{2.18}$$

Note that since $0 < v < c$ so $1 < \gamma\infty$.

2.1.3 Length contraction

One of the important consequences of special theory of relativity is that, when we go from one inertial frame to another, the length of a measuring rod changes. In order to see it, Let us consider a rod which is in rest in the reference frame S' and moving with velocity v along x-axis in the reference frame S. If the end points of the rod have coordinates (x_1, y, z) and (x_2, y, z)in the reference frame S and (x'_1, y, z) and (x'_2, y, z) in the reference frame S' then using Lorentz transformation

$$x'_1 = \gamma(x_1 - vt) \quad \text{and} \quad x'_2 = \gamma(x_2 - vt)$$

and so

$$x'_2 - x'_1 = \gamma(x_2 - x_1)$$

Note that $x'_2 - x'_1 = l_0$, is the length of the rod in the reference frame (S') in which the rod is rest so this length is called the proper length. If the length of the rod in S is l i.e., $x_1 - x_2 = l$ then:

$$\boxed{l = \frac{l_0}{\gamma} = l_0\sqrt{1 - v^2/c^2}}$$

Note that $\gamma > 1$ so $l < l_0$ and so the length of an object is always contracted when measured in a frame in which the object is moving.

2.1.4 Time dilation & Simultaneity

If we consider that the time time interval between two events which take place at time ${t'}_1$ and ${t'}_2$ when measured in the rest frame S' is τ_0 i.e., ${t'}_1 - {t'}_2 = \tau_0$ (which is called the proper time) then we can find the time interval τ between these two events in any arbitrary reference frame S with respect tow which our clocks are in motion.

$$ {t'}_1 = \gamma(t_1 - vx/c^2) \quad \text{and} \quad {t'}_2 = \gamma(t_2 - vx/c^2) \tag{2.19} $$

and so we can find that

$$ {t'}_1 - {t'}_2 = \gamma(t_1 - t_2) \quad \text{or} \quad t_1 - t_2 = ({t'}_1 - {t'}_2)\sqrt{1 - v^2/c^2} \tag{2.20} $$

or

$$ \boxed{\tau = \tau_0\sqrt{1 - v^2/c^2}} $$

Note that since $\tau < \tau_0$ so the time interval between two events is maximum in a reference frame in which the points at which these events are taking place do not move. Due to this increment in the time interval between two events when measured in a moving frame the clocks are slowed when observed in a moving frame. This phenomenon is called the time dilation. One of the interesting consequence of this is that if there are twins brothers one stay on earth and another goes in space then the person staying on earth will find that the aging has been slowed for his brother who has left for space.

The most important difference which special theory of relativity makes is the change in the notion of simultaneity: It is not an absolute notion. If two events are taking place at points x_1 and x_2 in the reference frame S at time t_1 and t_2 respectively and at ${x'}_1$ and ${x'}_2$ at time ${t'}_1$ and ${t'}_2$ respectively than

$$ {t'}_1 = \gamma(t_1 - vx_1/c^2); \quad \text{and} \quad {t'}_2 = \gamma(t_2 - vx_2/c^2) \tag{2.21} $$

and so

$$ {t'}_1 - {t'}_2 = \gamma(t_1 - t_2) - \gamma v(x_1 - x_2)/c^2 \tag{2.22} $$

From the above equation it is clear that if two events are simultaneous in the frame S i.e., $t_1 - t_2 = 0$ then they may not be simultaneous in the frame S' i.e., ${t'}_1 - {t'}_2 \neq 0$.

2.1.5 Doppler effect

One of the important consequences of time dilation is that when we observe the frequency of a light wave emitted by moving source then we find that this frequency is different from the frequency which was emitted by the source. This phenomenon is called the Doppler effect. This happens because we receive the same number of crests or troughs of wave in a time interval which is different from the time interval in which they were emitted.

Consider a source of light moving along x-axis with speed v and emits light of frequency ν. This source will emit $\nu d\tau_0$ number of waves in $d\tau_0$ time, however, since the source is moving so these waves will take longer time to reach. Particulerly in one second the source will advance by a distance of v mt, so light has to travel this extra distance and time taken by light to cover this extra distance will be v/c. Now the total number of waves emitted in $d\tau_0(1+v/c)$ time will be $d\tau_0(1+v/c)\nu$.

We will receive $d\tau_0(1+v/c)\nu$ number of pulses in $d\tau$ time interval so the number of pulse received in unit ν' time will be

$$\nu' = \frac{d\tau}{d\tau_0}\frac{1}{(1+v/c)}\nu$$

since we know

$$\frac{d\tau}{d\tau_0} = \frac{1}{\sqrt{1-v^2/c^2}}$$

so

$$\nu' = \nu\sqrt{\frac{1-v/c}{1+v/c}} \tag{2.23}$$

2.1.6 Velocity addition

In oder to see that how velocities are transformed when we go from one inertial frame to another consider an object moving along x-axis with the velocities $u = dx/dt$ and $u' = dx'/dt'$ in frames S and S' respectvely. From Lorentz transformations

$$x' = \gamma(x - vt)$$

$$t' = \gamma(t - vx/c^2)$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

From these equations

$$\frac{dx'}{dt} = \frac{(dx - vdt)}{(dt - xdx/c^2)} = \frac{(dx/dt - v)}{(1 - v(dx/dt)/c^2)}$$

or one can write

$$u' = \frac{u - v}{1 - uv/c^2} \tag{2.24}$$

Note that if an object is moving with the speed of light in the frame S i.e., $u = c$ then the above formula gives $u' = c$ or the speed of object does not change.

Here this is important that y and z component of velocity also get changed although the object is moving along x-axis. This is because velocities along any direction also depend on time. If we define the components of velocity along x, y and z direction by u_x, u_y and u_z then.

$$u'_y = \frac{dy'}{dt'} = \frac{dy/dt}{\gamma(1 - v(dx/dt)/c^2)} = \frac{v_y}{\gamma(1 - vu_x/c^2)}$$

This is true for the z component also

$$u'_z = \frac{dy'}{dt'} == \frac{v_z}{\gamma(1 - vu_x/c^2)}$$

2.1.7 Acceleration transformation

On the basis of velocity transformation laws one can find the acceleration of an object in a moving frame when its acceleration in the rest frame is give. Let us consider components of velocity of an object in frame S and S' are (u_x, u_y, u_z) and (u'_x, u'_y, u'_z) respectvely. Acceleration is defined as

$$a_x = \frac{du_x}{dt}; a_x = \frac{du_y}{dt}; a_x = \frac{du_z}{dt}$$

We know

$$u'_x = \frac{u_x - v}{1 - vu_x/c^2}$$

so

$$\frac{du'_x}{dt'} = \frac{du_x/dt'}{1 - vu_x/c^2} + (u_x - v)\frac{(u/c^2)(du_x/dt')}{(1 - vu_x/c^2)^2}$$

or

$$\frac{du'_x}{dt'} = \frac{(du_x/dt')}{(1 - vu_x/c^2)^2}(1 - v^2/c^2)$$

$$\frac{du'_x}{dt'} = (du_x/dt)\frac{(1 - v^2/c^2)}{(1 - vu_x/c^2)^2}dt/dt'$$

putting

$$\frac{dt'}{dt} = \frac{(1 - vu_x/c^2)}{\sqrt{1 - v^2/c^2}}$$

we get

$$a'_x = \frac{du'_x}{dt'} = a_x\frac{(1 - v^2/c^2)^{\frac{3}{2}}}{(1 - vu_x/c^2)^3} \tag{2.25}$$

and

$$a'_y = \frac{du'_y}{dt'} = a_y\frac{(1 - v^2/c^2)}{(1 - vu_x/c^2)^2} + \frac{v}{c^2}.a_x u_y\frac{(1 - v^2/c^2)}{(1 - vu_x/c^2)}$$

$$a'_z = \frac{du'_z}{dt'} = a_z\frac{(1 - v^2/c^2)}{(1 - vu_x/c^2)^2} + \frac{v}{c^2}.a_x u_z\frac{(1 - v^2/c^2)}{(1 - vu_x/c^2)}$$

Other components can be obtained by the same way.

2.1.8 Relativistic mass

Like length and time intervals, mass of an object also get changed when measured in a moving frame. In order to see how it change consider the following collision experiments.

There are two hard sphere A and B. The sphere A looks same in S' as B looks in S i.e.,

$$m'_A = m_B$$

If initial they are seprated by distance Y along y-axis and A is rest in frame S' and B in frame S. Now if through sphere A toward B along +y axis with speed v'_A and, B toward A along -y axis with speed v'_B with

$$v'_A = v_B$$

Now if after collision A moves along -y axis with speed v'_A and B along + y axis with speed v'_B then according to the conservation of momentum

$$m'_A v'_A = m'_B v'_B$$

where m'_B is the mass of B is S'. Since we know that $v'_A = v_B$ and $m'_A = m_B$ so this equation becomes

$$m_B v_B = m'_B v'_B$$

Now the problem is find how v_B and v'_B are related and this can be done as follows.

Observers in both frames will agree that A and B collide after A and B travels half the distance i.e., Y/2, however, they will not agree on time interval which the objects takes between their starting their journey and then reaching at the same position after the collision.

Let us consider τ_0 is round trip time for B in S'

$$\tau_0 = \frac{Y}{v'_B}$$

This time interval in frame S will be

$$\tau = \frac{Y}{v_B} = \tau_0 \frac{v'_B}{v_B}$$

or

$$v_B = v'_B \frac{\tau_0}{\tau}$$

Now from the expression of time dilation we know that

$$\frac{\tau_0}{\tau} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

so

$$v_B = v'_B \frac{1}{\sqrt{1 - v^2/c^2}}$$

putting this expression in the equation of conservation of momentum

$$m_B = m'_B \frac{1}{\sqrt{1 - v^2/c^2}}$$

Since m'_B is the mass of B in a frame in which it does not move so we call this mass the rest mass.

If the rest mass of an object is m_0 then its mass in a frame moving with velocity will be

$$m = m_0 \frac{1}{\sqrt{1 - v^2/c^2}} = m_0 \gamma \tag{2.26}$$

Since the multiplication factor γ is always greater then 1 so $m > m_0$.

This increase in mass is a consequence of the fact that the speed of an object can not increase indefinitely so when we supply energy to an object then in place of increasing velocity its mass increases.

2.1.9 Mass Energy equivalence

Consider we apply a force F of an object and it travels a distance ds in so the work done on the object will be Fds. In the absence of any dissipation this all work will be storder in the object in the form of Kinetic energy. After traveling a distance s under the action of force kinetic energy of the object will be

$$T = \int_0^s F ds = \int_0^s \frac{dp}{dt} ds = \int_0^s \frac{d(mv)}{dt} ds$$

or

$$T = \int_0^{mv} v d(mv) = \int_0^v v d\left(m_0 \frac{v}{\sqrt{1 - v^2/c^2}} \right)$$

Integrating by parts we get

$$T = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} - m_0 c^2 = mc^2 - m_0 c^2$$

This means that the total energy E of the particle

$$E = E_0 + T = m_o c^2 + T = mc^2$$

where $E_0 = m_0 c^2$ is the rest mass energy. So the total energy of the particle will be

$$E = mc^2 \tag{2.27}$$

Note that in special theory of relativity energy is considered as a time (zeroth) component of a four vector named four momentum.

$$p^\mu = (E, -pc) \quad \text{with} \quad p_\mu p^\mu = E^2 - p^2 c^2 = m_o^2 c^4 \tag{2.28}$$

So the total energy of a relativistic particle is given by the following expression

$$E = \sqrt{m_0 c^4 + p^2 c^2} \tag{2.29}$$

2.1.10 Four force and four momentum

As has been mentioned that the four momentum is defined as

$$p^\mu = (E, cp)$$

so is we differentiate it

$$\frac{dp^\mu}{dt} = \left(\frac{dE}{dt}, c\frac{dp}{dt} \right)$$

so the four force is

$$f^\mu = \frac{dp^\mu}{dt} = \left(\frac{dE}{dt}, c\frac{dp}{dt} \right)$$

Now using the fact

$$p = mv = \gamma m_0 v \quad \text{and} \quad \frac{dE}{dt} = \text{ power } = \gamma \, \mathbf{f}.\mathbf{v}$$

we find

$$f^\mu = \left(\gamma \frac{\mathbf{f}.\mathbf{v}}{c}, \gamma f \right) \tag{2.30}$$

2.2 Transformation of four vectors and tensors

In four dimensional Minkowski space position of an object is give by its four coordinates x^μ and corresponding this we have a vector with four components:

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

In special theory of relativity it is postulated that the form of physical laws should not change under the lorentz transformations. This means that right and left hand sides of mathematical expressions which represent physical laws should change by the same way. This is invoked in special theory of relativity by using some entities called the tensors which follow well defined laws of transformations under any general coordinate transformation.

Consider a general coordinate transformation:

$$x^\mu \longrightarrow x'^{\mu'} \tag{2.31}$$

Under this coordinate transformation any arbitrary tensor $T^{\alpha_1,\alpha_2,\alpha_3...\alpha_r}_{\beta_1,\beta_2,\beta_3...\beta_s}$ will transform by the following way:

$$T^{\alpha_1,\alpha_2,\alpha_3...\alpha_r}_{\beta_1,\beta_2,\beta_3...\beta_s} \longrightarrow T'^{\alpha_1,\alpha_2,\alpha_3...\alpha_r}_{\beta_1,\beta_2,\beta_3...\beta_s}$$

$$= \left(\frac{\partial x'^{\alpha_1}}{\partial x^{\mu_1}}\frac{\partial x'^{\alpha_2}}{\partial x^{\mu_2}}\frac{\partial x'^{\alpha_1}}{\partial x^{\mu_1}}....\frac{\partial x^{\alpha_r}}{\partial x^{\mu_r}}\right)\left(\frac{\partial x^{\nu_1}}{\partial x'^{\beta_1}}\frac{\partial x^{\nu_2}}{\partial x'^{\beta_2}}\frac{\partial x^{\nu_3}}{\partial x'^{\beta_1}}....\frac{\partial x^{\nu_s}}{\partial x'^{\beta_s}}\right)T'^{\mu_1,\mu_2,\mu_3...\mu_r}_{\nu_1,\nu_2,\nu_3...\nu_s} \tag{2.32}$$

Superscripts are called contravariant index and subscripts are called the covariant index. Here the tensor T is said be of contravariant rank of r and covariant rank of s. The main property of tensors is that two tensors of same type (same number of covariant and contravariant indices) transform by the same ways. In this chapter we will be mostly concerned with transformation of rank zero (scalars), one (vectors) and two.

Lorentz Scalars:

In special theory of relativity those quantities are said to be the lorentz scalars which do not transform under lorentz transformation. For example

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

is a lorentz scaler because it remain the same in all frames.

Lorentz vectors or four vectors:

In general any physical quantity like position, time, momentum, energy, force , work etc., can be represented by a components of a four vectors. It has been mention that position of an object in four dimensional space is represented by the four vectors x^μ. In general there are two type of for vectors named covariant represented by A_μ and contravarinet four vectors represented by A^μ. These two type of vectors follow different transformation laws which can be obtained by the transformation laws of general tensor T mentioned above.

$$A^\mu = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} \quad \text{and} \quad A_\mu = (A_0, A_1, A_2, A_3)$$

If we multiply A_μ with A^μ by the following way then we get a lorentz scalar quantity.

$$A_\mu A^\mu = (A_0, A_1, A_2, A_3) \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = A_0 A^0 - A_1 A^1 - A_2 A^2 - A_3 A^3$$

This type of multiplication is different from matrix multiplication, however, we can made it a matrix multiplication by using the following matrix $\eta_{\alpha\beta}$ which is called the metric

$$\eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.33}$$

Now in terms of this matrix

$$A_\mu A^\mu = \eta_{\mu\nu} A^\mu A^\nu$$

So the invariant line element is defined as

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

Under any general coordinate transformation $x^\mu \longrightarrow x'^\mu$ four vectors A^μ and A_μ transform by the following ways:

$$A'^\mu = T^\mu_\nu A^\nu$$

Where T^μ_ν is a 4×4 matrix which shows how the various components of the four vector A^μ will transform under the given coordinate transformations.

2.3 Lorentz Group

Group:

Any set $\{S\}$ of transformations with an operation (called the group multiplication) is said to be a group if it satisfies the following properties:

1. Closer:- If elements A and B are in $\{S\}$ then AoB should also be in $\{S\}$. Here o reprsents the group multiplication.

2. Identity:- There should exist an element I in the set which do not change an element when multiplied with it i.e., $IA = A$

3. Inverse:- Corresponding to every element A in $\{S\}$ there should exist an element A^{-1}, is called the inverse of A in $\{S\}$ such that $AoA^{-1} = I$.

4. Associative:- For any three elements A, B and C in $\{S\}$ the following relation should hold.

$$Ao(BoC) = (AoB)oC$$

There are many sets with certain operations which form groups. Any group can have finite or infinite number of elements. In former case we call the group discrete and in later case continuous or Lie group. Here we are mainly concerned about continuous groups. In what follows two important groups one rotation in three dimensional Eucledian space and another rotations in four dimensional Minkowski space are discussed.

2.3.1 Rotations

In order to understand group theoretical properties of lorentz transformation consider rotations in three dimensional Eucledian space.

In general, any vector in three dimensional space is give by a three vector

$$x^i = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Now if we rotate our coordinate system about the z-axis by θ angle then in new coordinate system the three vector x'^i will be

$$x'^i = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

This can also be expressed as

$$x'^i = R^i_j(\theta) x^j$$

One of the important properties of these type of rotations is that the length of vector does not change.

$$r^2 == x_i x^i = x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2 \tag{2.34}$$

Note that $x_i = x^{i^T}$ where x^{i^T} is the transformation of the vector x_i i.e., $x^{i^T} = (x, y, z)$. Now dropping indices

$$x' = Rx \quad \text{and} \quad x'^T = x^T R^T$$

so we can write

$$r' = x'^T x' = x^T R^T R x$$

Now from the equation (2.34) $r' = r$ or

$$x^T R^T R x = x^T x \quad \text{or} \quad R^T R = 1 \tag{2.35}$$

This is an important condition which says that matrices which represent rotations in three dimensional space should be orthogonal i.e., $R^T R = 1$. So the group formed by these transformation matrices R is called $O(3)$. We can have orthogonal groups in n-dimensional space also. There also same properties are followed and we call such groups $O(N)$.

If we take determinants of both sides of equation(2.35)

$$|R^T||R| = |R|^2 = 1$$

Now there are two possibilities: $|R| = +1$ and $|R| = -1$.

For the case $|R| = +1$ the matrices R form a group called the special orthogonal or SO e.g., $SO(3), SO(N)$.

So far we have considered rotation only about the z-axis and in this case a typical group element looks

$$R_z(\theta) = \begin{pmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

From this expression it is clear that $R_z(\theta)$ can take any value because θ is a continuous variable $< 0 < \theta < 2\pi$. One can easily show that $R_z(\theta)$ form a one parameter (which is θ) continuous or Lie group.

If we take θ very small then we find

$$R_z(\theta) = I + \theta \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + O(\theta^2)$$

Now we call the matrix M_Z which is defined as following the generators of this group

$$M_Z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

One can easily shows that any finite rotation matrix $R_z(\theta)$ can be obtained from the matrix M by the following way:

$$R_z(\theta) = e^{\theta . M_z} \tag{2.36}$$

We can also do same for the rotation about x and y-axis also and in these cases the rotation matrices are as follows:

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{pmatrix} \quad \text{and} \quad R_y(\theta) = \begin{pmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{pmatrix}$$

And corresponding infinitesimal generators are as follows:

$$M_x(\theta) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad M_y(\theta) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Now if we rotate our coordinate system first by θ_1 angle about x-axis followed by rotations θ_2 and θ_3 about the y and z-axis then the rotation matrix will look

$$R(\Theta) = R_x(\theta_1)R_y(\theta_2)R_z(\theta_3) = e^{\Theta . M}$$

where

$$\Theta.M = \theta_1 M_x + \theta_2 M_y + \theta_3 M_z$$

This is easy to show that the matrices M_i follow the following commutation relations:

$$[M_i, M_j] = \epsilon_{ijk} M_k \quad \text{where} \quad i = 1, 2, 3 \tag{2.37}$$

where

$$\epsilon_{ijk} = 1 \text{ for cyclic}, 0 \quad \text{for repeated, and} - 1 \quad \text{for anti-cyclic indices.}$$

2.3.2 Lorentz group

Any general coordinate transformation:

$$x^\mu \longrightarrow x'^\mu \quad \text{or} \quad x'^\mu = L^\mu_\nu x^\nu$$

is said to lorentz transformation if the following condition is satisfied

$${x_1}^2 - x_\mu x^\mu = {x'}_\nu x'^\nu \quad \text{or} \quad {x_0}^2 - {x_1}^2 - {x_2}^2 - {x_3}^2 \quad \text{remains constant.}$$

This means that matrices L^μ_ν must satisfy the following condition

$${x'}_\mu x'^\mu = [L^\mu_\nu x^\nu]^T L^\mu_\nu x^\nu = [x^\mu]^T x^\mu$$

or which shows that

$$L^T L = 1$$

This condition looks like the condition we have found for rotation, however, there is an important difference. In place of $x_0^2 + x_1^2 + x_2^2 + x_3^2$ as in ordinary rotations in four dimensional space we have $x_0^2 - x_1^2 + x_2^2 + x_3^2$ invariant.

Like in the case of rotation we can take detrminat of both side

$$|L|^2 = +1 \quad \text{which shows} \quad |L| = +1 \quad \text{or} \quad |L| = -1$$

We consider only the former case i.e., $|L| = +1$ and call it the proper lorentz transformation.

In order to find the form of matrices L let us consider lorentz transformation along x-axis.

$$ct' = \gamma(ct - vx/c)$$

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

Define the following new variables:

$$\beta = \frac{v}{z} \quad \text{where} \quad 0 < \beta < 1$$

$$\gamma = cosh\xi \quad \text{and} \quad \gamma\beta = sinh\ \xi \quad \text{which means} \quad \beta = tanh\xi$$

In terms of these variables lorentz transformation along x-axis can be written as follows:

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \\ x'^4 \end{pmatrix} = \begin{pmatrix} cosh\xi & -sinh\xi & 0 & 0 \\ -sinh\xi & cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ x'^4 \end{pmatrix}$$

or one can write

$$x'^\mu = L^\mu_\nu(\xi)x^\nu \quad \text{or} \quad x' = L_x(\xi)x \tag{2.38}$$

These transformation look like rotations in four dimensional but in x-t plane so to differentiate these from rotation we call these transformation **Boosts** along x-axis. These boosts along x-axis make a one parameter continuous group because the parameter ξ can take any value.

In case of finite boost along x-axis we have the following transformation matrix.

$$L_x(\xi) = \begin{pmatrix} cosh\xi & -sinh\xi & 0 & 0 \\ -sinh\xi & cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now if we take the limit $\xi \longrightarrow 0$ we get infinitesimal transformation matrix or generator.

$$K_x = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So in terms of this matrix one can express any finite lorentz boost along x-axis.

$$L_x(\xi) = e^{-\xi.K_x} \tag{2.39}$$

This all discussion follows for the boosts along y and z-axis also, however, in these cases we have the following transformation matrices and generators.

$$L_y(\xi) = \begin{pmatrix} cosh\xi & 0 & -sinh\xi & 0 \\ 0 & 1 & 0 & 0 \\ -sinh\xi & 0 & cosh\xi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad K_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$L_z(\xi) = \begin{pmatrix} cosh\xi & 0 & 0 & -sinh\xi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -sinh\xi & 0 & 0 & cosh\xi \end{pmatrix} \quad \text{and} \quad K_z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Now the fact is the Lorenz transformations along x, y and x-axis separately form one parameter continiuos groups, however, they all together do not form any group. This is because if we combine two lorentz boosts along different direction then they do not form of a boost.

Transformation matrix for β_1, β_2 and β_3 boosts along x, y and z axis respectvely is given by the following matrix:

$$L(tanh^{-1}\beta_1, tanh^{-1}\beta_2, tanh^{-1}\beta_3) =$$

$$\begin{pmatrix} \gamma & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ -\gamma\beta_1 & 1+(\gamma-1)\frac{\beta_1^2}{\beta^2} & (\gamma-1)\frac{\beta_1\beta_2}{\beta^2} & (\gamma-1)\frac{\beta_1\beta_3}{\beta^2} \\ -\gamma\beta_2 & (\gamma-1)\frac{\beta_1\beta_2}{\beta^2} & 1+(\gamma-1)\frac{\beta_2^2}{\beta^2} & (\gamma-1)\frac{\beta_2\beta_3}{\beta^2} \\ -\gamma\beta_3 & (\gamma-1)\frac{\beta_1\beta_3}{\beta^2} & (\gamma-1)\frac{\beta_2\beta_3}{\beta^2} & 1+(\gamma-1)\frac{\beta_3^2}{\beta^2} \end{pmatrix} \quad (2.40)$$

As has been mentioned that three dimensional lorentz transformation do not form a group, however, it has been found that if we include three dimensional rotations written in slightly form then these rotations and boosts form a six parameter continiuos group called the lorentz group.

$$R_x(\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & cos\theta & sin\theta \\ 0 & 0 & -sin\theta & cos\theta \end{pmatrix}; \quad R_y(\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & cos\theta & 0 & sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -sin\theta & 0 & cos\theta \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & cos\theta & sin\theta & 0 \\ 0 & -sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

And corresponding infinitesimal generators are as follows:

$$M_x(\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}; M_y(\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}; M_Z(\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now any proper lorentz transformation which involves a three dimensional rotation by Θ and three dimensional boost by Ξ can be written as follows:

$$L(\Theta, \Xi) = e^{-\Theta.M - \Xi.K} \quad (2.41)$$

Note that

$$\Theta.M = \theta_1 M_1 + \theta_2 M_2 + \theta_3 M_3$$

and

$$\Xi.K = \xi_1 K_1 + \xi_2 K_2 + \xi_3 K_3$$

where

$$\xi_1 = tanh^{-1}\beta_1, \xi_2 = tanh^{-1}\beta_2, \xi_3 = tanh^{-1}\beta_3,$$

It has been found that M and K satisfy the following commutation rules:

$$[M_i, M_j] = \epsilon_{ijk} M_k$$

$$[M_i, K_j] = \epsilon_{ijk} K_k$$

$$[K_i, K_j] = \epsilon_{ijk} M_k \quad (2.42)$$

2.3.3 Poincare group

In four dimensional space the following coordinate transformations are possible.

- Reflections:- These are given by the following transformations:

$$x^\mu \longrightarrow -x^\mu$$

  There are four such transformations corresponding to every dimensions.

- Translations:- These are given as

$$x^\mu \longrightarrow x^\mu + a^\mu$$

  These are also four, corresponding to every dimensions.

- Lorentz transformations:- Given as

$$x^\mu \longrightarrow \Lambda_{\mu\nu} x^\nu$$

If we include all these transformation then they form a 14 parameter group called the Poincare group.

2.4 Relativistic Electrodynamics

One of the main motivations behind special theory of relativity has been to find a class of transformations which keeps Maxwell's equations invariant. Maxwells equations are as follows:

$$\nabla . D = \rho \tag{2.43}$$

$$\nabla \times H = J + \frac{\partial D}{\partial t} \tag{2.44}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{2.45}$$

$$\nabla . B = 0 \tag{2.46}$$

Where $D = \epsilon_0 E$ and $B \mu_o H$.

It is possible to express E and B in terms of a vector A and scalar potential ϕ by the following way

$$E = -\nabla \phi + \frac{\partial A}{\partial t} \tag{2.47}$$

$$B = \nabla \times A \tag{2.48}$$

Note that same (E, B) can be expressed in terms of (ϕ', A') also, which are related to (ϕ, A) by the following way

$$\phi' = \phi - \frac{\partial \Lambda}{\partial t} \tag{2.49}$$

$$A' = A + \nabla\Lambda \tag{2.50}$$

These transformations are called the gauge transformations.

Maxwell's equations can be reduced in the following couple of equations:

$$\nabla^2\phi + \frac{\partial}{\partial t}(\nabla.A) = -\frac{\rho}{\epsilon_0} \tag{2.51}$$

and

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla\left(\nabla.A + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = -\mu_0 J \tag{2.52}$$

This is a coupled set of equation which can be decoupled by fixing the gauge (restricted gauge transformations). If we chose

$$\nabla.A + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0$$

Then the above equations become

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial\phi}{\partial t} = -\frac{\rho}{\epsilon_0} \tag{2.53}$$

and

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\mu_0 J \tag{2.54}$$

This pair of equations can be solved easily.

Now if make gauge transformations then in terms of new potential the condition required for decoupling becomes.

$$\nabla.A' + \frac{1}{c^2}\frac{\partial\phi'}{\partial t} = 0 = \nabla.A + \frac{1}{c^2}\frac{\partial\phi}{\partial t} + \nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} \tag{2.55}$$

This means that if

$$\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = 0$$

then under this transformation (ϕ, A) reamain lorentz invariant. So this condition is called the Lorentz corresponding gauge lorentz gauge. Note that there other gauge also like the Columb gauge i.e., $\partial A = 0$ etc.

Now we can write maxwell's equations in terms of four vectors. Define the following four vectors.

$$\partial^\mu = (\frac{\partial}{dt}, \nabla); \partial_\mu = (\frac{\partial}{dt}, -\nabla) \ \text{ and} = \partial_\mu\partial^\mu = \frac{\partial}{dt^2} - \nabla^2 = \Box$$

Define

$$J^\mu = (\rho/\epsilon_0, \mu_0 J) \quad \text{and} \quad A^\mu = (\phi, A) \tag{2.56}$$

Where J^μ and A^μ are called the four current and four potential respectvely. Not that

$$J_\mu = (\rho/\epsilon_0, -\mu_0 J) \quad \text{and} \quad A_\mu = (\phi, -A) \ \ \text{because} A_\mu = \eta_{\mu\nu}A^\nu$$

Now in term of these variables equation (2.53) and (2.54) can be written as follows.

$$\Box A^{\mu} = -J^{\mu} \tag{2.57}$$

In order to write Maxwell's equations in tensor (covariant) form define the following tensor:

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \tag{2.58}$$

This is called Electromagnetic field tensor. Note that this is an anti-symmetric tensor so

$$F_{\mu\nu} = -F_{\nu\mu}$$

Since we know that electric field intansity E

$$E = \frac{\partial A}{\partial t} - \nabla\phi$$

or we can write the i'th component as

$$E_i = \partial_0 A_i - \partial_i A_0 = F_{0i}$$

Magnetic field intansity B is as follows:

$$B = \nabla \times A$$

or

$$B_1 = \partial_2 A_3 - \partial_3 A_2; B_2 = \partial_3 A_1 - \partial_1 A_3; B_3 = \partial_1 A_2 - \partial_2 A_1$$

or in compact notations

$$B_i = \epsilon^{ijk} F_{jk} \tag{2.59}$$

We can write $F_{\mu\nu}$ in matrix form also.

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \tag{2.60}$$

This is useful to define one other tenour $\tilde{F}^{\mu\nu}$ called the duel tensor.

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & -E_X & 0 \end{pmatrix} \tag{2.61}$$

From this one can easily guess that the duel tensor can be obtained by

$$E \longrightarrow B \quad \text{and} \quad B \longrightarrow -E$$

Now all four Maxwell's equations can be represented by the following covariant equation

$$\partial^{\mu} F_{\mu\nu} = J_{\nu} \tag{2.62}$$

2.4.1 Lorentz transformation for E and B

Electromagnetic field tensor transforms as a second rank tensor under lorentz transformations so B and E also transforms according to that;

$$F'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}F_{\alpha\beta}$$

Note that lorentz transformations along x-axis are as follows:

$$x^0 = \gamma x'^0 + \beta\gamma x'^1; x^1 = \gamma x'^1 + \beta\gamma x'^2; x^2 = x'^2; x^3 = x'^3$$

So we can write

$$E'_i = F'_{0,i} = \frac{\partial x^\alpha}{\partial x'^0}\frac{\partial x^\beta}{\partial x'^i}F_{\alpha\beta} = \frac{\partial x^0}{\partial x'^0}\frac{\partial x^j}{\partial x'^i}F_{0j} + \frac{\partial x^j}{\partial x'^0}\frac{\partial x^k}{\partial x'^i}F_{jk}$$

This gives

$$E'_1 = E_1; E'_2 = \gamma(E_2 - \beta B_3); E'_3 = \gamma(E_3 + \beta B_2)$$

By the same way we can find the expression for B also

$$B'_1 = B1; B'_2 = \gamma(B_2 + \beta E_3); B'_3 = \gamma(B_3 - \beta E_2)$$

These expressions can be written in more general form

$$\mathbf{E}' = \gamma(\mathbf{E} + \beta \times \mathbf{B}) - \frac{\gamma^{\mathbf{2}}}{\gamma + \mathbf{1}}\beta(\beta.\mathbf{E}) \tag{2.63}$$

and

$$\mathbf{B}' = \gamma(\mathbf{B} - \beta \times \mathbf{E}) - \frac{\gamma^{\mathbf{2}}}{\gamma + \mathbf{1}}\beta(\beta.\mathbf{B}) \tag{2.64}$$

2.4.2 Covariant electrodynamics

Equation of motion in classical mechanics follows from the principle of least action which says that a classical particle follows that path which minimizes its action. Action of a classical particle is defined as

$$A = \int_{t_1}^{t_2} L dt \tag{2.65}$$

Where L is called the Lagrangian which id defined as follows

$$L = T - V \tag{2.66}$$

where T and V and the kinetic and potential energies of the particle. Or one can write

$$L = \frac{1}{2}m\dot{x}^2 - V(x)$$

In general L is a function of position x and its derivative $\dot{x}$ i.e., $L = L(x, \dot{x})$ so the action can be written as

$$A = \int_{t_1}^{t_2} L(x, \dot{x})dt \tag{2.67}$$

Now minimizing it

$$\delta A = 0 \ \text{ gives}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) - (\frac{\partial L}{\partial x}) \tag{2.68}$$

This is called Euler's Lagrangian equation and Newton's equation of motion directly follows from this.

$$\ddot{x} = -\frac{\partial V(x)}{\partial x}$$

or

$$\frac{dp}{dt} = -\frac{\partial V(x)}{\partial x}$$

Current density:

From the equation (2.57) it is clear that the source of electromagnetic field A^μ are four currents J^μ. For a system of charged particles one can express J^μ in the following form

$$J^\mu(x) = \int d\tau \sum_n e_n \delta^4(x - x_n(\tau)) U_n^\mu \tag{2.69}$$

Where τ is the proper time interval and $U_n^\mu = dx^\mu/d\tau$ is the four velocity of the i'th particle. This current shows the following continuity equation

$$\partial_\mu J^\mu = 0 \ \text{ which gives } \ \frac{\partial \rho}{\partial t} + \nabla.\mathbf{J} = 0 \tag{2.70}$$

In terms of this current density ordinary charge is defined as follows

$$Q = \int \rho d^3x = \int J^0(x) d^3x = \sum_n e_n$$

Where 'n' reprsents the index of a particle.

2.4.3 Lorentz force

Electromagnetic force acting on a charged particle is given by

$$f^\mu = eF^{\mu\nu}U_\nu \tag{2.71}$$

Taking its i'th component

$$f^i = \frac{dp^i}{dt} = eF^{i\nu}U_\nu = eF^{i0}U_0 + eF^{ij}U_j = e\,(E + v \times B)$$

This is the lorentz force expression. Equation of motion of a charged particle in electromagnetic field is given by the following equation

$$m\frac{d^2x^\mu}{d\tau^2} = \frac{e}{c}F^{\mu\nu}\frac{dx_\nu}{d\tau} \tag{2.72}$$

Note that

$$c^2 d\tau^2 = \eta^{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}ds^2$$

2.4.4 Energy momentum tensor

In covariant form energy and momentum are represented by a four vector called the four momentum.

$$p^\mu = (E, cP) \quad \text{and} \quad p_\mu = (E, -cP) \tag{2.73}$$

Note that

$$p^\mu p_\mu = E^2 - p^2 c^2$$

is a lorentz scaler quantity.

One can express p^μ for a system of particles in terms of the following momentum density

$$T^{\mu 0} = \sum\nolimits_n p_n^\mu(t)\delta^3(x - x_n(t))$$

Note that

$$\int T^{\mu 0} d^x = \sum p_n^\mu$$

Now define the current of four momentum p^μ as follows:

$$T^{\mu i} = \sum\nolimits_n p_n^\mu(t)\frac{dx_n^i}{dt}\delta^3(x - x_n(t))$$

Note we can write the expression for the four momentum density and four momentum current density by the following expression

$$T^{\mu\nu} = \sum\nolimits_n p_n^\mu(t)\frac{dx_n^\nu}{dt}\delta^3(x - x_n(t)) \tag{2.74}$$

using

$$p_n^\nu = E_n \frac{dx_n^\nu}{dt}$$

we find

$$T^{\mu\nu} = \sum\nolimits_n \frac{p_n^\mu p_n^\nu}{E_n}\delta^3(x - x_n(t)) \tag{2.75}$$

From this we can see that $T^{\mu\nu}$ is symmetric

$$T^{\mu\nu} = T^{\nu\mu}$$

We can also write $T^{\mu\nu}$ in a covariant form

$$T^{\mu\nu} = \sum\nolimits_n \int p_n^\mu \frac{dx_n^\nu}{d\tau}\delta^4(x - x_n(\tau)) \tag{2.76}$$

This tensor $T^{\mu\nu}$ follows the following continuity equation

$$\partial_\mu T^{\mu\nu} = G^\nu \tag{2.77}$$

Where G^ν is called the force density which is defined as follows:

$$G^\nu(x, t) = \sum\nolimits_n \delta^3(x - x_n(t))\frac{dp_n^\nu}{dt} \tag{2.78}$$

If the particles are free then

$$\partial_\mu T^{\mu\nu} = 0 \tag{2.79}$$

As it has been mentioned that for a particle in electromagnetic field

$$\frac{dp^\nu}{dt} = eF_\mu^{nu}\frac{dx^\mu}{dt}$$

So in the case on electromagnetic field

$$\partial_\mu T^{\mu\nu} = \sum\nolimits_n e_n \delta^3(x - x_n(t)) F_\mu^{nu}\frac{dx^\mu}{dt} = F_\mu^\nu J^\mu \tag{2.80}$$

From this expression it is clear that $T^{\mu\nu}$ is not a conserved quantity, however, we add the following extra term in $T^{\mu\nu}$ then we can get a conserved tensor.

$$T_{EM}^{\mu\nu} = F_\lambda^\mu F^{\nu\lambda} - \frac{1}{4}\eta^{\mu\nu} F_{\lambda\theta} F^{\lambda\theta} \tag{2.81}$$

Which gives

$$T_{EM}^{00} = \frac{1}{2}(\ \mathbf{E^2} + \mathbf{B^2}) \quad \text{and} \quad T_{EM}^{i0} = \frac{1}{2}(\ \mathbf{E}\times\mathbf{B})$$

Now we can see that

$$\partial_\mu T_{EM}^{\mu\nu} = -F_\mu^\nu J^\mu \tag{2.82}$$

and

$$\partial_\mu (T^{\mu\nu} + T_{EM}^{\mu\nu}) = 0$$

2.5 Classical fields

So far all the discussion was about point particles like electrons which can be labeled by the index n now we can apply same formalism for fields ϕ also. In order to do that follow the following prescriptions:

Replace position of particles by fields;

$$x_n \longrightarrow \phi_k(x)$$

Note that

$$n \longrightarrow x^\alpha, k$$

and

$$\dot{x}_n \longrightarrow \partial^\alpha \phi_k(x)$$

Now action for a system point particles is given by

$$A = \int \sum\nolimits_n L_n(x_n, \dot{x}_n) dt = \int L dt$$

with

$$L_n(x_n, \dot{x}_n) = \frac{1}{2} m \dot{x}_n^2 - V(x_n)$$

Not that we can write

$$A = \int \mathcal{L} d^4 x \quad \text{where} \quad L = \int \mathcal{L} d^3 x$$

In the case of fields in place of particle we have fields and in place of a running index α over all the particles for a system of particles there are fields $\phi^\alpha(x)$ where x is the argument of fields which varies continously and the index α runs over all the fields.

Lagrangian and Hamiltonian can be given in terms of the Lagrangian density $\mathcal{L}$ and Hamiltonian density $\mathcal{H}$.

$$L = \int \mathcal{L}(\phi_k, \partial^\alpha \phi_k) d^3 x \tag{2.83}$$

so the action is given by

$$A = \int d^4 x \mathcal{L}(\phi_k, \partial_\mu \phi_k)$$

Note if we vary action A

$$\delta A = 0$$

We get the equation of motion

$$\partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_k)} - \frac{\partial \mathcal{L}}{\partial \phi_k} = 0 \tag{2.84}$$

Example:

Take the following Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

So

$$\frac{\partial \mathcal{L}}{\partial^\mu \phi} = \partial_\mu \phi$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

so the equation of motion becomes

$$(\Box + m^2)\phi = 0$$

This equation is called the Klein Gordon equation.

Exercise:

Show that the Maxwell's equations which are given as

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

follows from the following Lagrangian density

$$\mathcal{L} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{c} J_\alpha A^\alpha$$

2.5.1 Perfect fluid

A perfect fluid is defined by the property that an observer moving with the fluid see that isotropic around him. Or one can say that for a perfect fluid the means free path of the collision between its particles is small then the length scales of observer.

This means in the frame S' which is moving with the fluid the energy momentum tensor is given by the following expression

$$T'^{ij} = p\delta'^{ij}; T'^{i0} = 0 \text{ and } T'^{00} = \rho \tag{2.85}$$

Using lorentz transformation this expression can be written in frame S in the following way

$$T^{ij} = p\delta^{ij} + (p+\rho)\frac{v^i v^j}{(1-v^2)}$$

$$T^{i0} = (p+\rho)\frac{v^i}{(1-v^2)}$$

$$T^{00} = \frac{(p+\rho v^2)}{(1-v^2)}$$

Note that here 'c=1', convention is followed. Here p and ρ are called energy and pressure density respectvely. These expression can be written in the following compact form.

$$T^{\mu\nu} = p\eta^{\mu\nu} + (p+\rho)U^\mu U^\nu$$

where

$$U^i = \gamma v \text{ and } U^0 = \gamma \text{ with } U_\mu U^\mu = -1 \tag{2.86}$$

2.5.2 Scalar field

For a scaler field ϕ Lagrangian density is defined as follows:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{2.87}$$

Energy momentum or stress energy tensor for $T_{\mu\nu}$ a scalar field with Lagrangian density $\mathcal{L}$ is defined by the following expression

$$T_{\mu\nu} = \partial_\mu\phi\partial_\mu\phi - \mathcal{L}\eta_{\mu\nu} \tag{2.88}$$

So for the above case

$$T_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)\eta_{\mu\nu}$$

Since $\eta_{\mu\nu} = (1,-1,-1,-1)$ so the various components of energy momentum tensor are as follows

$$T_{00} = \rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{2.89}$$

$$T_{ii} = p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{2.90}$$

Chapter 3

General Theory of relativity

3.1 Introduction

In order to keep the form of physical laws unchanged or *covariant* they must be written in form of *tensors* which follow some well defined rules under coordinate transformations. One can easily find various physical quantities in a "uniformly moving frame" once they have been given in a rest frame using lorentz transformations which are linear transformations. However, if we want to find physical quantities in a "non-uniformly moving" or an accelerated frame we need non-linear coordinate transformations, in order to keep the form of physical laws invariant.

Let us consider a system of massive particles in a uniform gravitational field which are interacting with each other. For such a system the equation of motion for a typical particle will look like:

$$m_i \frac{d^2 x_i}{dt^2} = m_i g + \sum_{j \neq i = 1}^{n} F_{ij} \tag{3.1}$$

Now if we make the following "non-linear coordinate transformation" then we can get rid of the gravity term.

$$x \longrightarrow x' = x - \frac{1}{2} g t^2$$

$$m_i \frac{d^2 {x'}_i}{dt^2} = \sum_{j \neq i = 1}^{n} {F'}_{ij} \tag{3.2}$$

From the above exercise we can not the following:

- In order to keep the form of physical expressions unchanged in a "non-uniformly moving" or an un accelerated frame we need non-linear coordinate transformations.

- Motion of a particle in a "uniform gravitational field" is equivalent to its motion in some "accelerated frame of reference". Sometime we call this statement the *weal equivalence principle also* and this can be generalized in the form of *strong equivalence principle*, which says "all physical laws in a uniform gravitational field are same as in an unaccelerated frame".

Note that one can remove the effect of gravitational field by going in an accelerated frame only when the gravitational field is uniform or same everywhere, however, one can always chose an infinitesimal small region of space in which gravitational field is constant and can replace gravitational by an accelerated frame.

3.1.1 Non-inertial frames of reference

All those coordinate frames which are related two each other by lorentz transformations are inertial frames. In these frames the four dimensional line elements has the following simple form:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = c^2 dt^2 - dx^2 - dy^2 - dz^2 \quad \text{where} \quad \eta_{\mu\nu} = diag(1, -1, -1, -1) \tag{3.3}$$

Special theory of relativity postulates that ds^2 is an "invariant physical quantity" under lorentz transformations. In this case components of $\eta_{\mu\nu}$ are constant or they have the same value at every space-time point. One interesting thing is that when we go from one inertial frame to another by lorentz transformation then the form of the line element do not change. It is always of the form of $c^2dt^2 - dx^2 - dy^2 - dz^2$. There are no cross terms like $dxdt$ etc., are present. However, when we go from some inertial to non-inertial frame then that this not the case. In order to see this let us go in a frame which is rotating about z-axis with uniform angular speed ω.

$$x' = xcos(\omega t) + ysin(\omega t)$$

$$y' = -xsin(\omega t) + ycos(\omega t)$$

$$z' = z, \quad \text{and} \quad t' = t$$

In this new frame line element will take the following form:

$$ds'^2 = c^2 dt^2 - dx'^2 - dy'^2 - dz'^2$$

$$ds'^2 = [1 - \omega^2(x^2 + y^2)]dt^2 - dx^2 - dy^2 - dz^2 - 2\omega y dx dt + 2\omega x dy dt \tag{3.4}$$

$$= g'_{00}(dx'^0)^2 + g'_{11}(dx'^1)^2 + g'_{22}(dx'^2)^2 + g'_{33}(dx'^3)^2 + +g'_{01}dx'^0 dx'^1 + g'_{02}dx'^0 dx'^2$$

From this expression one can note the following things:

- Here the form of line element get changed ie., it contains cross terms also and we can show that there does not exist any coordinate transformation by which we can transform it into usual Minkowski form.

- Metric components $g_{mu\nu}$ are not constant, they depend on space-time coordinates e.g., $g'_{00} = 1 - \omega^2(x^2 + y^2)$

3.1.2 Curved space-time

Wald define the curvature in the following three ways:

1. The failure of successive cpvarient derivatives of tensor field to commute.
2. The failure of parallel transport of a vector around an infinitesimally closed curve to return the vector to its original value.
3. The failure of initially parallel, infinitesimally nearby geodesics to remain parallel.

In the above case i.e., rotating frame, if we try to compute ratio of circumference to dimeter of a circle then it does not come π. This is because along the circumference length get contracted due to lorentz contraction, however, it remains the same along dimeter. From this one can conclude that when we go in a non-inertial frame then the geometry of space no longer remains flat or Eucledian it becomes curved.

3.1.3 Proper time

In order to find proper the time interval between two events we go in a frame in which these events take place simultaneously. Consider two space-time points 'A' and 'B' which have coordinates X^μ and $x^\mu + dx^\mu$ respectvely the line element will be

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{3.5}$$

Now we can go in some another coordinate system in which these two points are at the same spatial position

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = {g'}_{00} (dx'^0)^2 = c^2 dt'^2 = c^2 d\tau^2 \tag{3.6}$$

or

$$d\tau = dt \sqrt{\left(1 - \frac{v^2}{c^2}\right)} \quad \text{where} \quad v^2 = \frac{dx^2 + dy^2 + dz^2}{dt^2} \tag{3.7}$$

This formula allows us to compute the proper time interval between two points

$$\tau = \frac{1}{c} \int_{t_1}^{t_2} \sqrt{{g'}_{00}} \tag{3.8}$$

3.1.4 Synchronization of clocks

Four dimension interval between two space time point for ant general metric $g_{\mu\nu}$ can be decomposed in the following way

$$ds^2 = g_{00} (dx^0)^2 + g_{0i} dx^i dx^0 + g_{ij} dx^i dx^j$$

In order to synchronize clocks at these two points let us send a light pulse from 'A' to 'B' i.e., $ds^2 = 0$ which gives

$$(dx^0)^2 + \left[2\left(\frac{g_{0i}}{g_{00}}\right) dx^i\right] dx^0 + \left(\frac{g_{ij}}{g_{00}}\right) dx^i dk^j = 0$$

We get the following solutions of this quadratic equation

$$(dx^0)^{\pm} = \frac{1}{g_{00}} \left[-g_{0i}dx^i \pm \sqrt{\left(g_{0i}g_{0j} - g_{ij}g_{00}\right) dx^i dx^j} \right] \tag{3.9}$$

These two solutions are corresponding for the signal going from 'A' to 'B' and 'B' to 'A' respectvely. If 'B' receives the signal at time x^0 then this signal would has been emitted by 'A' at $x^0 + (dx^0)^+$ and this signal will return back to 'A' at $x^0 + (dx^0)^-$. We can compute the total time of journey

$$\Delta_{AB} = (dx^0)^+ - (dx^0)^- = \frac{2}{g_{00}} \sqrt{\left(g_{0i}g_{0j} - g_{ij}g_{00}\right) dx^i dx^j} \tag{3.10}$$

Corresponding to this time interval we can compute the proprt time interval

$$c^2 d\tau = dl^2 = g_{00} \left[(dx^0)^+ - (dx^0)^- \right] = \left(-g_{ij} + \frac{g_{0i}g_{0j}}{g_{00}} \right) dx^i dx^j \tag{3.11}$$

or

$$dl^2 = \gamma_{ij} dx^i dx^j \quad \text{with} \quad \gamma_{ij} = \left(-g_{ij} + \frac{g_{0i}g_{0j}}{g_{00}} \right)$$

If 'B' is the halfway between the moment of 'sending' and 'receiving' signal by 'A' then the time coordinate at 'B' will be

$$(x^0)^B = x^0 + \frac{1}{2} \left[(dx^0)^+ + (dx^0)^- \right] = x^0 - \frac{g_{0i}dx^0}{g_{00}} \tag{3.12}$$

This equation clearly shows that 'A' and 'B' can have same time coordinates or they can be synchronized if

$$g_{0i} = 0 \tag{3.13}$$

Now in this our metric take the following form

$$ds^2 = g_{00}(dx^0)^2 - g_{ij} dx^i dx^j$$

we can further choose that $g_{00} = 1$.

The coordinate system for which $g_{0i} = 0$ and $g_{00} = 1$ is called the synchronous coordinate system.

3.2 Mathematical terms

Some of the mathematical terms which are generally used in general theory of relativity are defined as follows:

1. **Manifold:-** Any n-space that looks locally R^n is called manifold, i.e., n-sphere (S^n), n-torus (T^n) etc.

2. **Mapping:-** A map ϕ is called one-to-one (or injective) if each element N has at most one element of M mapped into it. A map is called onto (or surjective) if each element of N ha at least one element of M mapped into it.

   Example:- $\phi R \longrightarrow R$: for $\phi(x) = e^x$ the map is one-to-one but for $\phi(x) = x^3 - x$ it is onto but not one to one, $\phi(x) = x^3$ is both (bijective) and $\phi(x) = x^2$ is neither.

   A map from R^m to R^n takes an m-tuple $(x^1, x^2, ..., x^m)$ to an n-tuple $(y^1, y^2, ..., y^n)$, and can therefore be thought of as a collection of n functions of ϕ^i of m variables

   $$y^n = \phi^n(x^1, x^2, ..., x^m)$$

   Any of these function is said to be C^p if it is Continious and p-times differentiable, if entire map $\phi : R^m \longrightarrow R^n$ is C^p is C^P if all of its components are at least C^p. C^0 map is continuous but not differentiable and C^∞ map is differentiable as many times as you like.

3. **Diffeomorphism:-** Two sets M and N are said to be diffeomorphic if there exists a C^∞ map $\phi : M \longrightarrow N$ with a C^∞ inverse $\phi^{-1} : M \longrightarrow N$.

4. **Homomorphism:-** Two spaces are homomorphic (topologically equivalent) if there is a Continious map between them with a Continious inverse. There exist spaces which are homomorphic but not diffeomorphic (topologically same but with distinct differentiable structure).

3.3 Metric tensor or the fundamental tensor

Four dimensional line elements is written in the following form:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{3.14}$$

Here $g_{\mu\nu}$ called the fundamental tensor or the metric tensor. It is second rank symmetric tensor i.e., $g_{\mu\nu} = g_{\nu\mu}$ so it has 10 components. This means that in general theory of relativity in place of a single gravitational potential (Newtonian potential) we have 10 "potentials". In any arbitrary gravitational these "potentials" or metric components are functions of space-time coordinates. As has been mentioned that for an infinitesimal region we can transform $g_{\mu\nu}$ in Minkowski form.

$$x^\mu \longrightarrow x'^\alpha$$

This gives

$$g'_{\alpha\beta} = \left(\frac{\partial x^\mu}{\partial x'^\alpha}\right)\left(\frac{\partial x^\nu}{\partial x'^\beta}\right) g_{\mu\nu} = \eta_{\alpha\beta}$$

Taking determinant of both sides

$$J^2 g = -1 \quad \text{or} \quad J = \frac{1}{\sqrt{-g}} \tag{3.15}$$

Here $J = \left|\frac{\partial x^\mu}{\partial x'^\alpha}\right|$ is the Jacobian of coordinate transformation and $g = |g_{\alpha\beta}|$.

Note that in general this expression is of the following form:

$$g' = J^2 g \quad \text{or} \quad \sqrt{g'} = J\sqrt{g} \tag{3.16}$$

1. Invariant volume:

Four dimensional volume is defined as follows:

$$d\tau = dx^0 dx^1 dx^2 dx^3 \tag{3.17}$$

In new coordinate system

$$d\tau' = dx'^0 dx'^1 dx'^2 dx'^3 = \left|\frac{\partial x'}{\partial x}\right| dx^0 dx^1 dx^2 dx^3 = \frac{1}{J} d\tau \tag{3.18}$$

From the equations (3.16) and (3.18)

$$\sqrt{g}' d\tau' = \sqrt{g} d\tau \tag{3.19}$$

Here $\sqrt{g}d\tau$ is a coordinate system independent quantity or inavarinet quantity so it is called the *inavarinet volume*. In general metric tensor $g_{\mu\nu}$ has the $(+,-,-,-)$ signatures so its determinant is a negative quantity. In order to find the invariant volume real, we define it by $\sqrt{-g}d\tau$

2. Tensor densities:

Any physical quantity for an infinitesimal volume can be written is terms of its four dimensional tensor density which does not depend on coordinate frames or is an invariant quantity.

$$D = \int_{\tau_1}^{\tau_2} \mathcal{D}\sqrt{-g}d\tau \tag{3.20}$$

Here $\mathcal{D}$ is the tensor density of the physical quantity D. From this expression it is clear that D is an invariant physical quantity because $\mathcal{D}$ and $\sqrt{-g}d\tau$ are invariant.

3.4 Mach principle

Inertial frames of reference are neither an absolute property of nature nor determined by the unobservable ether; they are determined relative to the motion of the rest of the matter in the universe.

3.5 Geometry

3.6 Non inertial reference frame

Special theory of relativity gives the rules for the transformations from a rest frame S to a uniformally moving frame S' or from one inertial frame to another inertial frame. It considers that the space-time is flat and the norm (A^{α}_{α}) of a four vector A_{α} remains invariant, and metric tensor $\nu_{\alpha\beta} = (-1, 1, 1, 1)$ remain same for all frames which are related by lorentz transformations. There are cases in which metric not only changes with space but it changes with time also, and this happens when a reference frame is no longer inertial or frame is accelerated.

$$d\tau^2 = g_{\alpha\beta} dx^{\alpha} dx^{\beta} \tag{3.21}$$

In most general form the metric $g_{\alpha\beta}$ is not fixed but it depends on the space-time coordinates like in the example given.

In a non inertial reference frame geometry of the space time is not flat but it is curved. Which can be understood easily in the case of rotating frame. If we measure the circumference and the dimeter in a rotating frame then their ratio is different from π because the yardstick which we are using for the length measurement get lorentz contracted along the circumference but remains invariant along the dimeter.

3.7 Principle of equivalence

Inertial mass of a body is defined as the ratio of an external force acting on it and the acceleration produced in it. Gravitational mass can be defined by same way except here we replace the external force by gravitation force. It has been observed that these two masses are almost equal up to a great accuracy. This observation motivate us to give a principle called the **equivalence principle** which says that the motion of a particle in gravitational field is same as in a non-inertial frame. This is the weak version of the equivalence principle. Strong equivalence principle say that all laws of the nature are same in a non-inertial frame and gravity.

> At every space time point in an arbitrary gravitational field it is possible to choose a **locally inertial coordinate system** such that, within a sufficiently small region of the point in question, the laws of nature takes the same form as in an unaccelerated Cartesian coordinate system in the absence of gravitation.

$$x \longrightarrow \xi$$

where x^{α} are coordinates in any general coordinate system and ξ^{α} are coordinates in a locally inertial coordinate system. All the effects of a gravity can be given by $\frac{\partial \xi^{\alpha}}{\partial x^{\mu}}$ Considering a freely falling coordinate system in a gravitational field, due to the equivalace principle, gravitational and inertial forces will cancels out.

$$\frac{d^2 \xi^{\alpha}}{d\tau^2} = 0 \tag{3.22}$$

This equation can be written in following form using the properties of the transformation from $\xi \longrightarrow x$

$$\frac{\partial^2 x^{\lambda}}{d\tau^2} + \Gamma^{\lambda}_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} \tag{3.23}$$

where

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \tag{3.24}$$

are called **Christofel's coefficients or connections** . These quantities are not tensors and they vanishes for a flat space time. We can always go in a coordinate system in which they vanish but their derivatives not these coordinates are called the **normal coordinates** Equation (3.23) is called the **geodesic equation** . Invariant time interval is given by

$$d\tau^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = \eta_{\alpha\beta} d\xi^{\alpha} d\xi^{\beta} \tag{3.25}$$

Where $n_{\alpha\beta} = (1, -1, -1, -1)$ is the metric of flat Minkowski space-time. This equation gives

$$g_{\mu\nu} = \frac{d\xi^{\alpha}}{\partial x^{\mu}} \frac{d\xi^{\beta}}{\partial x^{\nu}} \eta_{\alpha\beta} \tag{3.26}$$

differentiating this equation

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} = \frac{\partial^2 \xi^{\alpha}}{\partial x^{\lambda} \partial x^{\mu}} \frac{d\xi^{\beta}}{\partial x^{\nu}} \eta_{\alpha\beta} + \frac{\partial^2 \xi^{\beta}}{\partial x^{\lambda} \partial x^{\nu}} \frac{d\xi^{\alpha}}{\partial x^{\mu}} \eta_{\alpha\beta}$$

We can simplify this equation by using equation (3.24) and (3.26)

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} = \Gamma^{\sigma}_{\lambda\mu} g_{\sigma\nu} + \Gamma^{\rho}_{\lambda\nu} g_{\rho\mu}$$

writing other two permutations as following

$$\frac{\partial g_{\nu\lambda}}{\partial x^{\mu}} = \Gamma^{\sigma}_{\nu\mu} g_{\sigma\lambda} + \Gamma^{\rho}_{\lambda\mu} g_{\rho\nu}$$

$$\frac{\partial g_{\lambda\mu}}{\partial x^{\nu}} = \Gamma^{\sigma}_{\lambda\nu} g_{\sigma\mu} + \Gamma^{\rho}_{\mu\nu} g_{\rho\lambda}$$

Adding first two equations and subtracting third from it we get

$$\Gamma^{\rho}_{\lambda\mu} = \frac{1}{2} g^{\rho\nu} [\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\nu\lambda}}{\partial x^{\mu}} - \frac{\partial g_{\lambda\mu}}{\partial x^{\nu}}] \tag{3.27}$$

3.7.1 Transformation laws for $\Gamma^{\alpha}_{\beta\gamma}$ and covariant derivative

According to equation (3.27)

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{\partial x^{\alpha}}{\partial \xi^{\lambda}} \frac{\partial^2 \xi^{\lambda}}{\partial x^{\beta} \partial x^{\gamma}}$$

under the coordinate transformation $x^{\alpha} \longrightarrow x'^{\alpha}$ this quantity transforms as following

$$\Gamma'^{\alpha}_{\beta\gamma} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\rho}}{\partial x'^{\beta}} \frac{\partial x^{\sigma}}{\partial x'^{\gamma}} \Gamma^{\mu}_{\sigma\rho} + \frac{\partial x'^{\alpha}}{\partial x^{\sigma}} \frac{\partial^2 x^{\sigma}}{\partial x'^{\beta} \partial x'^{\gamma}}$$

$$\Gamma^{\mu}_{\sigma\rho} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x'^{\beta}}{\partial x^{\rho}} \frac{\partial x'^{\gamma}}{\partial x^{\sigma}} \Gamma'^{\alpha}_{\beta\gamma} - \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x'^{\beta}}{\partial x^{\rho}} \frac{\partial x'^{\gamma}}{\partial x^{\sigma}} \frac{\partial x'^{\alpha}}{\partial x^{\tau}} \frac{\partial^2 x^{\tau}}{\partial x'^{\beta} \partial x'^{\gamma}} \tag{3.28}$$

$\Gamma^{\alpha}_{\beta\gamma}$ does not transforms like a tensor so this quantity is not a tensor. We can also find the transformation rules for partial derivatives

$$\frac{\partial A^{\mu}}{\partial A^{\sigma}} = \frac{\partial}{\partial x^{\sigma}}[\frac{\partial x^{\mu}}{\partial x'^{\delta}} A'^{\delta}]$$

$$= \frac{\partial x^{\mu}}{\partial x'^{\delta}} \frac{\partial x'^{\rho}}{\partial x^{\sigma}} \frac{\partial A'^{\delta}}{\partial x'^{\rho}} + \frac{\partial^2 x^{\mu}}{\partial x'^{\delta} x'^{\rho}} \frac{\partial x'^{\rho}}{\partial x^{\sigma}} A'^{\delta} \tag{3.29}$$

we know that

$$A^{\rho} = \frac{\partial x^{\rho}}{\partial x'^{\epsilon}} A'^{\epsilon} \tag{3.30}$$

$$\frac{\partial A^{\mu}}{\partial x^{\sigma}} + \Gamma^{\mu}_{\sigma\rho} A^{\rho} = \frac{\partial x^{\mu}}{\partial x'^{\delta}} \frac{\partial x'^{\rho}}{\partial x^{\sigma}} \frac{\partial A'^{\delta}}{\partial x'^{\rho}} + \frac{\partial^2 x^{\mu}}{\partial x'^{\delta} x'^{\rho}} \frac{\partial x'^{\rho}}{\partial x^{\sigma}} A'^{\delta}$$

$$+ \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x'^{\beta}}{\partial x^{\rho}} \frac{\partial x'^{\gamma}}{\partial x^{\sigma}} \frac{\partial x^{\rho}}{\partial x'^{\epsilon}} \Gamma'^{\alpha}_{\beta\gamma} A'^{\epsilon} - \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x'^{\beta}}{\partial x^{\rho}} \frac{\partial x'^{\gamma}}{\partial x^{\sigma}} \frac{\partial x'^{\alpha}}{\partial x^{\tau}} \frac{\partial x^{\rho}}{\partial x'^{\epsilon}} \frac{\partial^2 x^{\tau}}{\partial x'^{\beta} \partial x'^{\gamma}} A'^{\epsilon}$$

after interchanging α, δ and γ, ρ third term becomes

$$\frac{\partial x^{\mu}}{\partial x'^{\delta}} \frac{\partial x'^{\beta}}{\partial x^{\gamma}} \frac{\partial x'^{\rho}}{\partial x^{\sigma}} \frac{\partial x^{\gamma}}{\partial x'^{\epsilon}} \Gamma'^{\delta}_{\beta\rho} A'^{\epsilon} = \frac{\partial x^{\mu}}{\partial x'^{\delta}} \frac{\partial x'^{\rho}}{\partial x^{\sigma}} \Gamma'^{\delta}_{\beta\rho} A'^{\beta}$$

forth term

$$= \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x'^{\beta}}{\partial x^{\rho}} \frac{\partial x'^{\gamma}}{\partial x^{\sigma}} \frac{\partial x'^{\alpha}}{\partial x^{\tau}} \frac{\partial x^{\rho}}{\partial x'^{\epsilon}} \frac{\partial^2 x^{\tau}}{\partial x'^{\beta} \partial x'^{\gamma}} A'^{\epsilon} = \delta^{\mu}_{\tau} \delta'^{\beta}_{\epsilon} \frac{\partial x'^{\gamma}}{\partial x^{\sigma}} \frac{\partial^2 x^{\tau}}{\partial x'^{\beta} \partial x'^{\gamma}} A'^{\epsilon}$$

$$= \frac{\partial x'^{\gamma}}{\partial x^{\sigma}} \frac{\partial^2 x^{\mu}}{\partial x'^{\beta} \partial x'^{\gamma}} A'^{\beta}$$

This term cancels with the second term. Finally we have

$$\frac{\partial A^{\mu}}{\partial x^{\sigma}} + \Gamma^{\mu}_{\sigma\rho} A^{\rho} = \frac{\partial x^{\mu}}{\partial x'^{\delta}} \frac{\partial x'^{\rho}}{\partial x^{\sigma}} [\frac{\partial A'^{\delta}}{\partial x'^{\rho}} + \Gamma'^{\delta}_{\beta\rho} A'^{\beta}] \tag{3.31}$$

This equation can be written as

$$D_{\sigma} A^{\mu} = A^{\mu}_{;\sigma} = \frac{\partial x^{\mu}}{\partial x'^{\delta}} \frac{\partial x'^{\rho}}{\partial x^{\sigma}} A'^{\delta}_{;\rho}$$

$$D_{\sigma} A^{\mu} = A^{\mu}_{;\sigma} = \frac{\partial A^{\mu}}{\partial x^{\sigma}} + \Gamma^{\mu}_{\sigma\rho} A^{\rho} \tag{3.32}$$

The quantity $A^{\mu}_{;\sigma}$ transforms like a second rank tensor and called **covariant derivative** . Covariant derivative for higher rank tensors can be find by same way

$$A^{\mu\nu}_{\rho;\sigma} = \partial_{\sigma} A^{\mu\nu}_{\rho} + \Gamma^{\mu}_{\alpha\sigma} A^{\alpha\nu}_{\rho} + \Gamma^{\nu}_{\alpha\sigma} A^{\alpha\mu}_{\rho} - \Gamma^{\alpha}_{\rho\sigma} A^{\mu\nu}_{\alpha} \tag{3.33}$$

3.8 Parallel transport

Moving a vector/tensor from one point to another while keeping it constant is called the **parallel transport** . In a flat space we can move a vector from one point to another without being changed it but we can not carry out this operation in curved space. In curved space the result of parallel transport will depend on the path taken.

Let us take a curve $x^\mu(\lambda)$, where λ is parameter joining to points A and B, now if a tensor T is parallel transported along this curve then the condition

$$\frac{dT}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial T}{\partial x^\mu} = 0 \tag{3.34}$$

must be satisfied.

Covariant differentiation along the curve

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda}D_\mu$$

taking the covariant derivative of the tensor along the curve

$$\frac{D}{d\lambda}T^{\mu_1,\mu_2,\ldots\ldots\mu_k}_{\nu_1,\nu_2,\ldots.\nu_l} = \frac{dx^\mu}{d\lambda}D_\mu T^{\mu_1,\mu_2,\ldots\ldots\mu_k}_{\nu_1,\nu_2,\ldots.\nu_l} = 0$$

for a vector V^μ

$$\frac{DV^\mu}{d\lambda} = \frac{dx^\nu}{d\lambda}D_\nu V^{mu} = \frac{dx^\nu}{d\lambda}\left[\frac{\partial V^\mu}{\partial x^\nu} + \Gamma^\mu_{\sigma\nu}V^\sigma\right] = 0$$

or

$$\frac{\partial V^\mu}{\partial\lambda} + \Gamma^\mu_{\sigma\nu}\frac{\partial x^\nu}{\partial\lambda}V_\sigma = 0 \tag{3.35}$$

This equation is called the equation of parallel transport.

If we take $V^\mu = \frac{\partial x^\mu}{\partial\lambda}$ which is the four velocity then we get

$$\frac{\partial^2 x^\mu}{\partial\lambda^2} + \Sigma^\mu_{\nu\sigma}\frac{\partial x^\nu}{\partial\lambda}\frac{\partial x^\sigma}{\partial\lambda} \tag{3.36}$$

This equation is nothing but the **Geodesic equation** .

3.9 Newtonian limit

If a particle is moving slowly then we can find its geodesic equation in following approximating

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{3.37}$$

where $h_{\alpha\beta} << \eta_{\alpha\beta}$ from equation 3.23

$$\frac{\partial^2 x^i}{d\tau^2} + \Gamma^i_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} = 0 \tag{3.38}$$

$$\frac{\partial^2 t}{d\tau^2} = 0 \tag{3.39}$$

$$\Gamma^i_{00} = -\frac{1}{2}\eta^{i\beta}\frac{\partial h_{00}}{\partial x^\beta} = -\frac{1}{2}\bigtriangledown h_{00}$$

$$\frac{\partial^2 x^i}{d\tau^2} = \frac{1}{2}\bigtriangledown h_{00}(\frac{dt}{d\tau})^2$$

using (3.30) we get

$$\frac{\partial^2 x^i}{dt^2} = \frac{1}{2}\bigtriangledown h_{00}$$

but Newton's law says

$$\frac{\partial^2 x^i}{dt^2} = -\bigtriangledown h_{00}\Phi$$

from last two equations we get

$$h_{00} = -2\phi + constant; g_{00} = -(1+2\phi) \tag{3.40}$$

Gravitational redshift General four dimensional line element can be written in the following way

$$ds^2 = g_{00}dt^2 + g_{0i}dx^i dt + g_{ij}dx^i dx^j$$

where $i, j = 1, 2, 3$ In a weak gravitational field general line element

$$d\tau^2 = (1+2\phi)dt^2 - dx^2 - dy^2 - dz^2$$

Let us consider a wave pulse is emitted at x_1 at time t_1 and this pulse is received at x_2 at time t_2 and next pulse is emitted at x_1 at time $t_1 + dt_1$ and received at x_2 at time $t_2 + dt_2$.

At point x_1 these two pulses will be seprated by the proper time interval

$$d\tau^2 = (1 + 2\phi(x_1))dt_1^2$$

and at x_2

$$d\tau^2 = (1 + 2\phi(x_2))dt_2^2$$

since $d\tau$ is an invariant interval so

$$(1 + 2\phi(x_1))dt_1^2 = (1 + 2\phi(x_2)dt_2^2$$

or

$$\frac{dt_1}{dt_2} = \sqrt{\frac{(1+2\phi(x_2))}{(1+2\phi(x_1))}}$$

$$\frac{dt_1}{dt_2} = \frac{(1+\phi(x_2))}{(1+\phi(x_1))} = \frac{\nu_2}{\nu_1} = \frac{\lambda_1}{\lambda_2}$$

or

$$\frac{\lambda_1}{\lambda_2} - 1 = \frac{(1+\phi(x_2))}{(1+\phi(x_1))} - 1$$

$$\frac{\lambda_1 - \lambda_2}{\lambda_2} \approx \frac{\phi(x_2) - \phi(x_1)}{\phi(x_1)} = \frac{d\phi}{\phi}$$

$$\frac{d\lambda}{\lambda} = -\frac{d\phi}{\phi} \tag{3.41}$$

When we go from a region of high gravitational field to a region of low gravitational field then the wavelength of light signal increases or frequency decreases , since this shifts is toward the lower end of spectrum so it is called the **gravitational reds shift**

3.10 Curvature tensor

We know that

$$D_\beta \xi_\alpha = \partial_\beta \xi_\alpha - \Gamma^\sigma_{\alpha\beta}\xi_\sigma$$

so

$$D_\gamma D_\beta \xi_\alpha = \partial_\gamma \partial_\beta \xi_\alpha - (\partial_\lambda \xi_\alpha)\Gamma^\lambda_{\beta\gamma} - (\partial_\beta \xi_\lambda)\Gamma^\lambda_{\alpha\gamma} - (D_\gamma \xi_\sigma)\Gamma^\sigma_{\alpha\beta} - \xi_\sigma D_\gamma(\Gamma^\sigma_{\alpha\beta})$$

or

$$D_\gamma D_\beta \xi_\alpha = \partial_\gamma \partial_\beta \xi_\alpha - (\partial_\lambda \xi_\alpha)\Gamma^\lambda_{\beta\gamma} - (\partial_\beta \xi_\lambda)\Gamma^\lambda_{\alpha\gamma} - (\partial_\gamma \xi_{\sigma - \xi_\lambda \Gamma^\lambda_{\sigma\gamma}})\Gamma^\sigma_{\alpha\beta} -$$

$$\xi_\sigma[\partial_\gamma(\Gamma^\sigma_{\alpha\beta}) + \Gamma^\sigma_{\lambda\gamma}\Gamma^\lambda_{\alpha\beta} - \Gamma^\lambda_{\alpha\gamma}\Gamma^\sigma_{\lambda\beta} - \Gamma^\lambda_{\beta\gamma}\Gamma^\sigma_{\alpha\lambda}] \tag{3.42}$$

and

$$D_\beta D_\gamma \xi_\alpha = \partial_\beta \partial_\gamma \xi_\alpha - (\partial_\lambda \xi_\alpha)\Gamma^\lambda_{\gamma\beta} - (\partial_\gamma \xi_\lambda)\Gamma^\lambda_{\alpha\beta} - (\partial_\beta \xi_{\sigma - \xi_\lambda \Gamma^\lambda_{\sigma\beta}})\Gamma^\sigma_{\alpha\gamma} -$$

$$\xi_\sigma[\partial_\beta(\Gamma^\sigma_{\alpha\gamma}) + \Gamma^\sigma_{\lambda\beta}\Gamma^\lambda_{\alpha\gamma} - \Gamma^\lambda_{\alpha\beta}\Gamma^\sigma_{\lambda\gamma} - \Gamma^\lambda_{\gamma\beta}\Gamma^\sigma_{\alpha\lambda}] \tag{3.43}$$

Subtracting equation (7) from equation (6) we get

$$D_\gamma D_\beta \xi_\alpha - D_\beta D_\gamma \xi_\alpha = -[\partial_\gamma \Gamma^\sigma_{\alpha\beta} - \partial_\beta \Gamma^\sigma_{\alpha\gamma} + \Gamma^\lambda_{\alpha\beta}\Gamma^\sigma_{\lambda\gamma} - \Gamma^\lambda_{\alpha\gamma}\Gamma^\sigma_{\lambda\beta}]\xi_\sigma$$

this equation also can be written as

$$[D_\gamma, D_\beta] = -R^\sigma_{\alpha\beta\gamma}\xi_\sigma \tag{3.44}$$

where $R^\sigma_{\alpha\beta\gamma}$ is called *curvature tensor* it is defined as following

$$R^\sigma_{\alpha\beta\gamma} = \partial_\gamma \Gamma^\sigma_{\alpha\beta} - \partial_\beta \Gamma^\sigma_{\alpha\gamma} + \Gamma^\lambda_{\alpha\beta}\Gamma^\sigma_{\lambda\gamma} - \Gamma^\lambda_{\alpha\gamma}\Gamma^\sigma_{\lambda\beta} \tag{3.45}$$

we can write curvature tensor if following form also

$$R_{\lambda\alpha\beta\gamma} = g_{\sigma\lambda}R^\sigma_{\alpha\beta\gamma}$$

Properties of curvature tensor

Some important properties of curvature tensor are listed below.

- *antisymmetric properties* .

  $R_{\lambda\alpha\beta\gamma}$ is antisymetric in first two indices λ and α and also in last two indicesβ and γ .

- *symmetric properties* .

  $R_{\lambda\alpha\beta\gamma}$ is symetric when first two indices λ and α are interchanged with last two indicesβ and γ .

- *Number of components* .

  we know due to antisymmetry we have $\frac{n(n-1)}{2}$ permutations for λ, α and $\frac{n(n-1)}{2}$ for β, γ now due to symmetry of first and last pair total number of permutation of all four indices

$$= \frac{(\frac{n(n-1)}{2})(\frac{n(n-1)}{2} + 1)}{2}$$

  for four dimensional space we get total number of nonzero components

$$= \frac{(\frac{4(4-1)}{2})(\frac{4(4-1)}{2} + 1)}{2} = 21$$

  so in four dimensional space we have 21 non zero elements.

- *Bianchi identity*

$$D_\alpha R_{\beta\gamma\delta\sigma} + D_\beta R_{\gamma\alpha\delta\sigma} + D_\gamma R_{\alpha\beta\delta\sigma} = 0 \tag{3.46}$$

  Proof

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^\lambda_{\sigma\mu\nu}$$

  In normal coordinates $\Gamma = 0, \partial\Gamma \neq 0$

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda}[\partial_\mu \Gamma^\lambda_{\sigma\nu} - \partial_\nu \Gamma^\lambda_{\sigma\mu}]$$

  where

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

  with simplification

$$R_{\rho\sigma\mu\nu} = \frac{1}{2} g_{\rho\lambda} g^{\lambda\tau}[\partial_\mu\partial_\sigma g_{\tau\nu} - \partial_\mu\partial_\tau g_{\sigma\nu} - \partial_\nu\partial_\sigma g_{\tau\mu} + \partial_\nu\partial_\sigma g_{\sigma\mu}]$$

$$= \frac{1}{2}[\partial_\mu\partial_\sigma g_{\rho\nu} - \partial_\mu\partial_\rho g_{\sigma\nu} - \partial_\nu\partial_\sigma g_{\rho\mu} + \partial_\nu\partial_\rho g_{\sigma\mu}]$$

  taking its covariant derivative

$$R_{\rho\sigma\mu\nu;\lambda} = \frac{1}{2}[\partial_\lambda\partial_\mu\partial_\sigma g_{\rho\nu} - -\partial_\lambda\partial_\mu\partial_\rho g_{\sigma\nu} - \partial_\lambda\partial_\nu\partial_\sigma g_{\rho\mu} + \partial_\lambda\partial_\nu\partial_\rho g_{\sigma\mu}]$$

Now changing $\rho \longrightarrow \sigma \longrightarrow \lambda \longrightarrow \rho$

$$R_{\sigma\lambda\mu\nu;\rho} = \frac{1}{2}[\partial_\rho\partial_\mu\partial_\lambda g_{\sigma\nu} - -\partial_\rho\partial_\mu\partial_\sigma g_{\lambda\nu} - \partial_\rho\partial_\nu\partial_\lambda g_{\sigma\mu} + \partial_\rho\partial_\nu\partial_\sigma g_{\lambda\mu}]$$

and

$$R_{\lambda\rho\mu\nu;\sigma} = \frac{1}{2}[\partial_\sigma\partial_\mu\partial_\rho g_{\lambda\nu} - -\partial_\sigma\partial_\mu\partial_\lambda g_{\rho\nu} - \partial_\sigma\partial_\nu\partial_\rho g_{\lambda\mu} + \partial_\sigma\partial_\nu\partial_\lambda g_{\rho\mu}]$$

Adding these three equations

$$R_{\rho\sigma\mu\nu;\lambda} + R_{\sigma\lambda\mu\nu;\rho} + R_{\lambda\rho\mu\nu;\sigma} = 0 label C20 \tag{3.47}$$

This identity is called **Bianchi identity**

- *Ricci tensor and Ricci scalar*

  Ricci tensor is defined as

$$R_{\alpha\beta} = R^{\lambda}_{\alpha\lambda\beta}$$

  and Ricci scalar is defined as

$$R = g_{\alpha\beta}R^{\alpha\beta}$$

  a new tensor called *Einstein tesor* $G_{\alpha\beta}$ can be defined in terms of Ricci tesor and Ricci scalar.

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}$$

3.11 Tensor densities

Any physical quantity which transforms like a scaler in coordinate transformation except by a multiplication factor is called the tensor density. Two examples of tensor density are as following.

- determinant of $g_{\mu\nu}$

  Under the coordinate transformation $x \longrightarrow x'$

$$g'^{\mu\nu} = \frac{\partial x^\mu}{\partial x'^\sigma}\frac{\partial x^\nu}{\partial x'^\rho}g^{\sigma\rho}$$

  or

$$|g'| = \left|\frac{\partial x}{\partial x'}\right|^2 |g|$$

  From this equation it is clear that the g transforms like a scaler except the multiplication by Jacobian $J^2 = \left|\frac{\partial x}{\partial x'}\right|^2$. Since there are two powers of J so g is called a tensor density of weight -2.

  we can take $g'^{\sigma\rho} = \eta^{\sigma\rho}$ and taking the determinant of both sides

$$-1 = J^2|g|$$

  which gives $J = \frac{1}{\sqrt{-g}}$

- Invariant volume

  Four dimensional volume d^4x transforms by following way under general coordinate transformation

$$d^4x' = \left|\frac{\partial x'}{\partial x}\right| d^4x$$

  Which is a tensor density of weight -1.

  If we take $\sqrt{g'}d^4x'$ then it does not transforms

$$\sqrt{g'}d^4x' = \left|\frac{\partial x}{\partial x'}\right|\left|\frac{\partial x'}{\partial x}\right|\sqrt{g}d^4x = \sqrt{g}d^4x$$

  So $\sqrt{-g}d^4x$ is called invariant volume element.

3.12 Einstein's Field equations

1.Vacuum equations

In the absence of any matter total energy density of vacuum is contributed by gravitation or curvature so the Lagrangian density will be

$$\mathcal{L} = \frac{R}{8\pi G}$$

So the corresponding action, which is called the Hilbert action is as follows:

$$S_H = \frac{1}{8\pi G}\int d^nx\sqrt{-g}\mathcal{L} = \int d^nx\sqrt{-g}R = \int d^nx\sqrt{-g}R_{\mu\nu}g^{\mu\nu}$$

Now applying the variational principle

$$\delta S_H = 1\ over 8\pi G\int d^nx\left[\sqrt{-g}R_{\mu\nu}(\delta g^{\mu\nu}) + \sqrt{-g}(\delta R_{\mu\nu})g^{\mu\nu} + (\delta\sqrt{-g})R_{\mu\nu}g^{\mu\nu}\right] = 0$$

$$= \delta S_1 + \delta S_2 + \delta S_3$$

In order to compute δS_2 we have to find $\delta R_{\mu\nu}$ which can be find as follows. we know

$$R^\rho_{\mu\lambda\nu} = \partial_\lambda\Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\sigma\lambda}\Gamma^\sigma_{\mu\nu} - \partial_\nu\Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\sigma\nu}\Gamma^\sigma_{\mu\lambda} \tag{3.48}$$

we can compute

$$\Delta_\lambda\delta\Gamma^\rho_{\mu\nu} = \partial_\lambda\delta\Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\sigma\lambda}\delta\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\lambda}\delta\Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\mu\lambda}\delta\Gamma^\rho_{\nu\sigma} \tag{3.49}$$

interchanging λ and ν

$$\Delta_\nu\delta\Gamma^\rho_{\mu\lambda} = \partial_\nu\delta\Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\sigma\nu}\delta\Gamma^\sigma_{\mu\lambda} - \Gamma^\sigma_{\lambda\nu}\delta\Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\mu\nu}\delta\Gamma^\rho_{\lambda\sigma} \tag{3.50}$$

subtracting equation (3.50) from equation (3.49)

$$\Delta_\lambda\delta\Gamma^\rho_{\mu\nu} - \Delta_\nu\delta\Gamma^\rho_{\mu\lambda} = \partial_\lambda\delta\Gamma^\rho_{\mu\nu} - \partial_\nu\delta\Gamma^\rho_{\mu\lambda} + (\Gamma^\rho_{\sigma\lambda}\delta\Gamma^\sigma_{\nu\lambda} + \delta\Gamma^\rho_{\sigma\lambda}\Gamma^\sigma_{\nu\lambda}) - \left(\Gamma^\sigma_{\mu\lambda}\delta\Gamma^\rho_{\nu\sigma} + \delta\Gamma^\sigma_{\mu\lambda}\Gamma^\rho_{\nu\sigma}\right)$$

$$= \partial_\lambda \delta\Gamma^\rho_{\mu\nu} - \partial_\nu \delta\Gamma^\rho_{\mu\lambda} + \delta\left(\Gamma^\rho_{\sigma\lambda}\Gamma^\sigma_{\nu\lambda}\right) - \delta\left(\Gamma^\sigma_{\mu\lambda}\Gamma^\rho_{\nu\sigma}\right)$$
$$= \delta\left[\partial_\lambda \Gamma^\rho_{\mu\nu} - \partial_\nu \delta\Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\sigma\lambda}\Gamma^\sigma_{\nu\lambda} - \Gamma^\sigma_{\mu\lambda}\Gamma^\rho_{\nu\sigma}\right]$$

$$= \delta R^\rho_{\mu\lambda\nu} \tag{3.51}$$

Now

$$\delta(R_{\mu\nu}) = \delta R^\lambda_{\mu\lambda\nu} = \partial_\lambda \delta\Gamma^\lambda_{\mu\nu} - \partial_\nu \delta\Gamma^\lambda_{\mu\lambda}$$

and

$$\delta(R_{\mu nu})g^{\mu\nu} = \Delta_\sigma\left(g^{\mu\nu}\delta\Gamma^\sigma_{\mu\nu} - g^{\mu\sigma}\delta\Gamma^\lambda_{\mu\lambda}\right)$$

now

$$\delta S_2 = \frac{1}{8\pi G}\int d^n x\sqrt{-g}\Delta_\sigma\left(g^{\mu\nu}\delta\Gamma^\sigma_{\mu\nu} - g^{\mu\sigma}\delta\Gamma^\lambda_{\mu\lambda}\right) = 0$$

By Stokes theorem.

Now in order to compute the third term δS_3 we need to compute $\delta\sqrt{-g}$.

Using the formula

$$\text{Trace}\,(lnM) = ln\,(\text{Det}M) \tag{3.52}$$

or we can differentiate it

$$\text{Trace}\left(M^{-1}\delta M\right) = \frac{\delta\ \text{Det}M}{\text{Det}M}$$

Now take

$$M = g^{\mu\nu} \quad \text{So} \quad M^{-1} = g_{\mu\nu}, \quad \text{Det } g_{\mu\nu} = g^{-1} \quad \text{Where} \quad g = \text{Det}g^{\mu\nu}$$

Now we have

$$\frac{\delta(g^{-1})}{g^{-1}} = g_{\mu\nu}\delta g^{\mu\nu}$$

or

$$\delta(g^{-1}) = g^{-1}g_{\mu\nu}\delta g^{\mu\nu}$$

Now

$$\delta(\sqrt{-g}) = \delta\left[(-g^{-1})^{\frac{-1}{2}}\right] = \frac{-1}{2}(-g^{-1})^{\frac{-3}{2}}\delta(g^{-1}) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$$

Now our remaining integrle is

$$\delta S_H = \delta S_1 + \delta S_3 = \frac{1}{8\pi G}\int d^n x\sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]\delta(g^{\mu\nu}) = 0 \tag{3.53}$$

This implies that

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \tag{3.54}$$

2. Matter

In the presence of matter we have the following Lagrangian density

$$\mathcal{L} = \frac{R}{8\pi G} + \mathcal{L}_{\text{Matter}}$$

and in this case

$$\delta S = \delta S_H + \delta S_{\text{Matter}} = 0$$

one can write

$$\delta S_{\text{Matter}} = \int d^n x \sqrt{-g} T_{\mu\nu} \delta(g^{\mu\nu})$$

where $T_{\mu\nu}$ is the energy-momentum tensor for matter.

Now we have

$$\delta S = \int d^n x \sqrt{-g} \left[R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + 8\pi G T_{\mu\nu} \right] \delta(g^{\mu\nu}) = 0$$

So we get

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{3.55}$$

or in dimensional form

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} \tag{3.56}$$

3.13 solution of Einstein's Equations

3.14 Non-Eucedian Geometry

General theory of relativity is a geometric theory of gravity. From the observations it has been proved that light travels along curved path in presence of a massive body. One of the examples is the bending of star light when it passes close to Sun. This was first time observed in 1919 and was the first experimental test of general theory of relativity. Gravitational lensing, Black holes are also the examples of the bending of light. If we assume the curving of space by massive body in place of bending of light then all the phenomenon mentioned above can be explained. Gravitational effects of a massive body can be represented by the deformation of local space time geometry produced by the body. In the absence of gravity geometry of space time is Eucedian but when there is gravity it becomes Non Eucedian. Few of the differences between Eucledian and non-Eucledian geometry are as following.

- In a flat or Eucledian space for a triangle

  $$A + B + C = \pi$$

  Where A, B, C a are the interior angles.

  and for a circle

  $$\frac{C}{r} = 2\pi$$

  Where C and r are the circumference and radius respectvely.

- For a triangle in non-Eucledian space

  $$A + B + C = \pi + \frac{S}{a^2}$$

  and for a circle

  $$\frac{C}{r} = 2\pi \frac{sin\frac{r}{a}}{\frac{r}{a}}$$

  here S is the area and a is the curvature of space.

3.15 Maximally symmetric spaces

If any space is symmetric in one coordinate system then it may not be symmetric in another for example sphere is spherical symmetric only when we chose spherical polar coordinates otherwise not .So we can conclude that any particular symmetry of a physical problem is manifested only when 'right' kind of coordinate system is used .Such symmetries are coordinate dependent.In most of the problems we need a *covariant* or coordinate independent idea of symmetry . *Killing vectors* serves this purpose.

3.15.1 Isometry

Any contuinious transformation which keeps metric $g_{\mu\nu}$ *form invariant* is called *isometry* .

For any coordinate transformation

$$x^{\mu} \longrightarrow \tilde{x}^{\acute{\mu}} \tag{3.57}$$

$$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) \tag{3.58}$$

We know for a general coordinate transformation.

$$\tilde{g}_{\acute{\mu}\acute{\nu}}(\tilde{x}) = \frac{\partial x^{\mu}}{\partial \tilde{x}^{\acute{\mu}}}\frac{\partial x^{\nu}}{\partial \tilde{x}^{\acute{\nu}}} g_{\mu\nu}(x) \tag{3.59}$$

equation (3) also can be written as

$$g_{\mu\nu}(x) = \frac{\partial \tilde{x}^{\acute{\mu}}}{\partial x^{\mu}}\frac{\partial \tilde{x}^{\acute{\nu}}}{\partial x^{\nu}} = \tilde{g}_{\acute{\mu}\acute{\nu}}(\tilde{x})$$

3.15.2 Killing equation

Let us consider

$$\tilde{x}^{\acute{\mu}} = x^{\acute{\mu}} + \epsilon\xi^{\acute{\mu}} \tag{3.60}$$

where ϵ is an infinitesimally small quantity and $\xi^{\acute{\mu}}$ is the generator of transformations.

equation (4) gives

$$\frac{\partial \tilde{x}^{\acute{\mu}}}{\partial x^{\mu}} = \delta^{\acute{\mu}}_{\mu} + \epsilon\partial_{\mu}\xi^{\acute{\mu}} + O(\epsilon^2)$$

$$\frac{\partial \tilde{x}^{\acute{\nu}}}{\partial x^{\nu}} = \delta^{\acute{\nu}}_{\nu} + \epsilon\partial_{\nu}\xi^{\acute{\nu}} + O(\epsilon^2)$$

$$\tilde{g}_{\acute{\mu}\acute{\nu}}(\tilde{x}) = \tilde{g}_{\acute{\mu}\acute{\nu}}(x) + \epsilon\xi^{\lambda}\partial_{\lambda}\tilde{g}_{\acute{\mu}\acute{\nu}}(x) + O(\epsilon^2)$$

putting these values in equation (3)

$$g_{\mu\nu}(x) = (\delta^{\acute{\mu}}_{\mu} + \epsilon\partial_{\mu}\xi^{\acute{\mu}})(\delta^{\acute{\nu}}_{\nu} + \epsilon\partial_{\nu}\xi^{\acute{\nu}})(\tilde{g}_{\acute{\mu}\acute{\nu}}(x) + \epsilon\xi^{\lambda}\partial_{\lambda}\tilde{g}_{\acute{\mu}\acute{\nu}}(x))$$

or

$$g_{\mu\nu}(x) = \delta^{\acute{\mu}}_{\mu}\delta^{\acute{\nu}}_{\nu}\tilde{g}_{\acute{\mu}\acute{\nu}}(x) + \epsilon[\delta^{\acute{\mu}}_{\mu}\partial_{\nu}\xi^{\acute{\nu}}\tilde{g}_{\acute{\mu}\acute{\nu}}(x) + \delta^{\acute{\nu}}_{\nu}\partial_{\mu}\xi^{\acute{\mu}}\tilde{g}_{\acute{\mu}\acute{\nu}}(x) + \delta^{\acute{\mu}}_{\mu}\delta^{\acute{\nu}}_{\nu}\xi^{\lambda}\partial_{\lambda}\tilde{g}_{\acute{\mu}\acute{\nu}}(x)]$$

or

$$g_{\mu\nu}(x) = \tilde{g}_{\mu\nu}(x) + \epsilon[\delta^{\acute{\mu}}_{\mu}\partial_\nu \xi^{\acute{\nu}} \tilde{g}_{\acute{\mu}\acute{\nu}}(x) + \delta^{\acute{\nu}}_{\nu}\partial_\mu \xi^{\acute{\mu}} \tilde{g}_{\acute{\mu}\acute{\nu}}(x) + \delta^{\acute{\mu}}_{\mu}\delta^{\acute{\nu}}_{\nu}\xi^\lambda \partial_\lambda \tilde{g}_{\acute{\mu}\acute{\nu}}(x)]$$

from (2)

$$g_{\mu\nu}(x) = \tilde{g}_{\mu\nu}(x)$$

so

$$\delta^{\acute{\mu}}_{\mu}\partial_\nu \xi^{\acute{\nu}} \tilde{g}_{\acute{\mu}\acute{\nu}}(x) + \delta^{\acute{\nu}}_{\nu}\partial_\mu \xi^{\acute{\mu}} \tilde{g}_{\acute{\mu}\acute{\nu}}(x) + \delta^{\acute{\mu}}_{\mu}\delta^{\acute{\nu}}_{\nu}\xi^\lambda \partial_\lambda \tilde{g}_{\acute{\mu}\acute{\nu}}(x) = 0$$

or

$$\partial_\nu \xi^{\acute{\nu}} \tilde{g}_{\mu\acute{\nu}}(x) + \partial_\mu \xi^{\acute{\mu}} \tilde{g}_{\acute{\mu}\nu}(x) + \xi^\lambda \partial_\lambda g_{\mu\nu}(x) = 0$$

or

$$\partial_\nu[\xi^{\acute{\nu}} g_{\mu\acute{\nu}}(x)] + \partial_\mu[\xi^{\acute{\mu}} g_{\acute{\mu}\nu}(x)] - \xi^{\acute{\nu}}\partial_\nu g_{\mu\acute{\nu}}(x) - \xi^{\acute{\mu}}\partial_\nu g_{\acute{\mu}\nu}(x) + \xi^\lambda \partial_\lambda g_{\mu\nu}(x) = 0$$

or

$$\partial_\nu \xi_\mu + \partial_\mu \xi_\nu - \xi^\lambda[\partial_\nu g_{\mu\lambda} + \partial_\mu g_{\nu\lambda} - \partial_\lambda g_{\mu\nu}] = 0$$

or

$$\partial_\nu \xi_\mu + \partial_\mu \xi_\nu - 2\xi^\lambda g_{\lambda\sigma}\Gamma^\sigma_{\mu\nu} = 0$$

or

$$[\partial_\nu \xi_\mu - \xi_\sigma \Gamma^\sigma_{\mu\nu}] + [\partial_\mu \xi_\nu - \xi_\sigma \Gamma^\sigma_{\mu\nu}] = 0$$

or

$$D_\nu \xi_\mu + D_\mu \xi_\nu = 0 \tag{3.61}$$

where D_ν represent covariant derivative. Equation (5) is called *killing equation* .

Chapter 4

Smooth Universe

4.1 Introduction

Einstein's equations are used to study the dynamics of the universe. They allow us to obtain the form of space-time metric $g_{\mu\nu}$ for a given matter distribution, which is characterized by energy momentum tensor $T_{\mu\nu}$. In general, it is not possible to solve Einstein's equations for an arbitrary case, however, if we use symmetry properties or make some assumptions then they can be solved.

There are two important assumptions named the **Weyl's postulates** and the **cosmological principle** which provide some information about the space-time metric.

4.1.1 Weyl's Postulate

In special theory of relativity time is considered as the forth dimension of a four dimensional space called the Minkowski space. In this space trajectories of material particles are represented by world lines or geodesics. If we draw a constant time surface in this four dimensional space then the surface will be perpendicular to the world lines of all objects or galaxies in the universe. Whenever two or more objects collide with each other, their world lines intersect. Weyl's postulate makes the following statement: " **The world lines of galaxies form a 3-bundle of non-intersecting geodesics orthogonal to a series of space like hypersurfaces.**"

In general a typical world line is given by

$$x^{\mu} = \text{constant}$$

A constant time hypersurface or spacelike hypersurface is given by

$$x^{0} = \text{constant}$$

Orthogonality of the Weyl's postulate implies that $g_{0i} = 0$. Now we can solve the geodesic equations on a spacelike hypersurface and use the above conditions to guess the form of space-time metric.

$$\frac{d^2x^{\mu}}{ds^2} + \Gamma^{\mu}_{\nu\lambda}\frac{\partial x^{\nu}}{ds}\frac{\partial x^{\lambda}}{ds} = 0$$

For spatial part

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{\nu\lambda}\frac{\partial x^\nu}{ds}\frac{\partial x^\lambda}{ds} = 0$$

This gives $\Gamma^i_{00} = 0$ and this means that

$$\frac{\partial g_{00}}{\partial x_i} = 0 \quad \text{or} \quad g_{00}(x^\mu) = g_{00}(x_0)$$

g_{00} components depends on time only, so we can normalize it to unity. Weyl's postulate gives the following form of space-time metric

$$ds^2 = c^2dt^2 - \gamma_{ij}(x^l)dx^idx^j$$

4.1.2 Cosmological principle

There is no reason to believe that there is a preferred place or direction in the universe. This belief is based on many observational evidences like the distribution of galaxies and cosmic microwave background radiation (CMBR) temperature on the sky. This assumption is stated in the form of a postulate called the cosmological principle which says: " **On an average matter distribution in the universe at large scales (> 100 Mpc) is homogeneous and isotropic or at any point of time the universe looks the same from all places and along all directions.** "

Note that the universe looks homogeneous and isotropic only for a preferred class of observers called the *fundamental observers*. Mathematically cosmological principle translates into the following form of energy momentum tensor:

$$T^{\mu\nu}(\vec{x}, t) = T^{\mu\nu}(t) \quad \text{and} \quad T^{0i} = T^{i0} = 0 \tag{4.1}$$

For a perfect fluid this comes in the following form:

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}$$

Where ρ and p are the density and pressure of the fluid respectvely.

Note that homogeneity and isotropy are two different concepts they do not mean the same thing i.e., we can have an anisotropic and homogeneous universe but not an isotropic and inhomogeneous universe. One can show that if the universe is isotropic about two spatial points then it must be homogeneous also.

Homogeneity means that the physical quantities relavent to cosmology remain invariant under spatial translations and isotropy means they remain invariant under rotations. There is a principle called the perfect cosmological principle which postulates that the universe does not only look the same along all directions and from all places at some time, it looks so for all times. This means that the universe does not evolve in time. Big Bang theory does not accept this principle.

4.1.3 Hubble's law

In order to keep the average appearance (smeared out matter distribution) of the universe the same, at every point and along every directions, galaxies, which are the building blocks of the universe can not move in arbitrary ways, they follow a law called the Hubble's law.

Let us consider three galaxies labeled 1, 2 and 3, in a homogeneous and isotropic universe which are at positions $\vec{x}_1$, $\vec{x}_2$ and $\vec{x}_3$ and have velocities $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$. Now one can think that their velocities are related to their positions by some universal law (which we have to find here) i.e., $\vec{v}_1 = v(\vec{x}_1), \vec{v}_2 = v(\vec{x}_2)$ and $\vec{v}_3 = v(\vec{x}_3)$.

If we chose the origin of our coordinate system at the position of first galaxy then the position of the second galaxy will be $\vec{x}_2 - \vec{x}_1$, and according to our chosen law its velocity $\vec{v}_{21}$ will be $v(\vec{x}_2 - \vec{x}_1)$. Homogeneity allows us to chose the origin of coordinate system at any point we wish and this means that

$$\vec{v}_{21} = \vec{v}(\vec{x}_2 - \vec{x}_1) = \vec{v}(\vec{x}_2) - \vec{v}(\vec{x}_1)$$

The only form of v which satisfies this is as follows:

$$\vec{v}(\vec{x}) = \quad \text{constant} \quad \times \vec{x} \quad \text{or} \quad v = Hx$$

Here H is a constant called the *Hubble constant* and the above law is called the Hubble's law. We can write Hubble law in the following form also:

$$v_i = H_{ij} x^j \quad \text{where} \quad i = 1, 2, 3 \quad \text{and} \quad H \quad \text{is a diagonal matrix}$$

In a homogeneous and isotropic universe H is just a function of time this means that at any point of time the value of H remains the same everywhere in the universe.

From Hubble law it is clear that the unit of the Hubble constant is $\frac{1}{time}$, however, in cosmology we use another unit i.e., km/sec/Mpc. This is because the velocities of galaxies are measured in term of km/sec and distances in Mpc. The value of Hubble's constant is uncertain by a large amount so in place of using its exact value, we write it in the following form.

$$H = 100h \text{ km/sec/Mpc} \quad \text{where} \quad 0 < h < 1$$

4.2 Non-relativistic cosmology

4.2.1 Newtonian cosmology

Many of the interesting features of our universe can be explained on the basis of Newtonian mechanics. General theory of relativity comes into picture only when either we are studying scales smaller than the Schwardchild radius or larger than the horizon size.

In order to study the dynamics of the universe let us consider a sphere of radius R, mass M, and uniform density ρ at any instant of time, call it t. It is known in classical mechanics that the dynamics of a massive object within a sphere will not be affected by the matter distribution outside the sphere.

In a homogeneous and isotropic universe Hubble law is the only permitted law for velocities

$$v(t) = H(t)r$$

This equation predicts that the distance between two particles A and B will change as

$$r_{AB}(t) = r_{AB}(t_0)exp\left[\int_{t_0}^{t} H(t)dt\right]$$

for a uniform distribution of matter

$$\rho(t) = \frac{3M}{4\pi R^3(t)}$$

this gives

$$\frac{d\rho}{dt} = -3\frac{3M}{4\pi R^3(t)}\frac{1}{R(t)}\frac{dR(t)}{dt} = -3\rho H$$

equation of motion

$$\frac{d^2R}{dt^2} = -\frac{GM}{R^2}$$

from the equation of motion

$$\frac{d}{dt}(HR) = R\frac{dH}{dt} + H\frac{dR}{dt} = R\frac{dH}{dt} + H^2R = -\frac{4\pi}{3}G\rho R$$

or

$$\frac{dH}{dt} = -H^2 - -\frac{4\pi}{3}G\rho \tag{4.2}$$

we know that

$$\frac{dH}{dt} = \frac{d}{dt}\left(\frac{\dot{R}}{R}\right) = -H^2q - H^2$$

where $q = -\frac{\ddot{a}a^2}{\dot{a}^2}$ is called the *deacceleration parameter*. From the above two equations

$$q = \frac{4\pi G\rho}{3H^2} = \frac{\Omega}{2}$$

taking the first integral of the equation of motion

$$\frac{1}{2}\left(\frac{dR}{dt}\right)^2 - \frac{GM}{R} = \frac{1}{2}\left(\frac{dR}{dt}\right)^2 - \frac{4\pi G\rho}{3}R^2 = A$$

or

$$\frac{\dot{R}^2}{R^2} - \frac{A}{R^2} = \frac{8\pi G\rho}{3} \tag{4.3}$$

Here A is a constant which depends on the total energy and can be calculated by boundary conditions (BC). Here we can take $t = t_0, \rho = \rho_0, R = R_0, H = H_0$

$$A = R_0^2H_0^2\left(\frac{8\pi G\rho_0}{3H_0^2} - 1\right)$$

or

$$A = R_0^2 H_0^2 \left(\frac{\rho_0}{\rho_c} - 1 \right) = R_0^2 H_0^2 \left(\Omega_0 - 1 \right)$$

ρ_c and Ω are called the *critical density* and the *density parameter respectvely.* Dynamics of the universe is governed by equation (4.3) and (4.2) which are are called the *Friedmann's equations.* Geometry of the universe depends on the value ofΩ which reprsents the total energy density of the universe. We measure Ω and A by observations.

4.2.2 Geometry of space

In order to understand the geometrical structure of our three dimensional space let us consider a four dimensional Eucledian space which is characterized by the following metric:

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 = d\sigma^2 + dx_4^2 \tag{4.4}$$

Now let us consider a three sphere S^3 of radius R embedded in this space.

$$R^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 = r^2 + x_4^2 \quad \text{here} \quad r < R \quad \text{and} \quad r^2 = x_1^2 + x_2^2 + x_3^2 \tag{4.5}$$

This equation gives

$$dx_4^2 = \frac{r^2 dr^2}{R^2 - r^2}$$

putting this in equation (4.4) we get

$$dl^2 = d\sigma^2 + \frac{r^2 dr^2}{R^2 - r^2}$$

We can use spherical polar coordinates

$$\left.\begin{array}{c} x_1 = r \sin\theta \cos\phi \\ x_2 = r \sin\theta \sin\phi \\ x_3 = r \cos\theta \end{array}\right\}$$

so

$$d\sigma^2 = dx_1^2 + dx_2^2 + dx_3^2 = d\sigma^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2$$

Now our metric becomes

$$dl^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + \frac{r^2 dr^2}{R^2 - r^2} = \frac{R^2 dr^2}{R^2 - r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

We can write the full space-time metric

$$ds^2 = dt^2 - dl^2$$

here $c = 1$ is used.

$$ds^2 = c^2 dt^2 - \left[\frac{dr^2}{1 - \frac{r^2}{R^2}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right]$$

Let us take

$$\frac{1}{R^2} = \frac{k}{R_0^2} \quad \text{where} \quad k = +1$$

and divide and multiply the whole equation by R_0^2

$$ds^2 = dt^2 - R_0^2 \left[\frac{d(r/R_0)^2}{1 - k(r/R_0)^2} + (r/R_0)^2 (d\theta^2 + \sin^2\theta d\phi^2) \right]$$

This equation can be written in terms of a dimensionless variable $r' = r/R_0$

$$ds^2 = dt^2 - R_0^2 \left[\frac{dr'^2}{1 - kr'^2} + r'^2 (d\theta^2 + \sin^2\theta d\phi^2) \right]$$

Now we can replace r' by r but we will keep in mind that r is actually r' which is a dimensionless variable.

$$ds^2 = dt^2 - R_0^2 \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right]$$

In this equation 'k' is a dimensionless quantity, howere, R_0 carries the dimension of length and it is called the scale factor. In this case we can always chose the value of R_0 such that 'k=1'.

$$ds^2 = dt^2 - R_0^2 \left[\frac{dr^2}{1 - r^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right]$$

Note that we have the condition $r \leq 1$. Let us write r in terms of a new variable χ i.e., $r = \sin\chi$. In this new variable spatial section of our metric becomes

$$dl^2 = R_0^2 [d\chi^2 + \sin^2\chi (d\theta^2 + \sin^2\theta d\phi^2)]$$

We know that our coordinates are bounded by

$$0 \leq \chi \leq \pi; 0 \leq \theta \leq \pi; 0 \leq \phi \leq 2\pi$$

Now it easy to find the volume of this spatial section which is the surface of a S^3 embedded in R^4

$$V_3 = \int_0^\pi R_0 d\chi \int_0^\pi R_0 \sin\chi d\theta \int_0^{2\pi} R_0 \sin\chi \ \sin\theta d\phi = 2\pi^2 R_0^3$$

This all discussion was about the case in which a three sphere S^3 was embedded in a four dimensional Eucledian space R^4. We know that this is not only the case: we can have a Hyperboloidal three surface H^3 embedded in a four dimensional Eucledian space R^4. For this case

$$-R^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 = r^2 + x_4^2$$

It is fairly easy to show that for this case our metric comes in the following form

$$ds^2 = dt^2 - R_0^2 \left[\frac{dr^2}{1 + r^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right]$$

Again if we define $r = \sinh\chi$ then it can be written as

$$dl^2 = R_0^2[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)]$$

here coordinates are bounded by

$$0 \le \chi \le \infty; 0 \le \theta \le \pi; 0 \le \phi \le 2\pi$$

From the above equation it is clear that in this case volume of the spatial section is infinite. In both the above cases i.e., S^3 and H^3 there was a scale provided by the curvature of three space, however, we can also have a case in which curvature of the spatial section is infinite or the space is flat or $dx_4 = 0$ or $x_4 = constant$. In this case we have a R^3 surface embedded in R^4, it is easy that for this case the metric comes in the following form.

$$ds^2 = dt^2 - R_0^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

Let us again define $r = \chi$ and compute the volume of the three section

$$V_3 = \int_0^1 R_0 d\chi \int_0^\pi R_0\chi d\theta \int_0^{2\pi} R_0\chi\ \sin\theta d\phi = \frac{4}{3}\pi R_0^3$$

In this discussion we considered three special case, however, we can have any three surface of arbitrary shape embedded in four dimensional space. In general, these surfaces can be one of the following three types: positively curved (like S^3), negatively curved (like H^3) and flat (R^3). In order to understand the physical meaning of this let us consider a a case of two dimensional surface which is embedded in a three dimensional Eucledian space. In this case it is easy to see that the ratio of circumference to dimeter of a circle drawn on this surface is π only when the surface is globally flat.

In three dimensional universe one can find the curvature of space by counting number of galaxies (N) brighter than a certain brightness (flux f) in spatial volumes of different size assuming that all galaxies are equally bright and they are distribute uniformly. We know that in a flat space if we are at a distance r from an object which has intrinsic luminosity then the flux received by us will be

$$f = \frac{L}{4\pi r^2}$$

or

$$r = \sqrt{\frac{L}{4\pi f}}$$

If we count the number of objects N in a volume of radius of r then

$$N = n \times \frac{4\pi r^3}{3}$$

where n is the spatial density of objects. From the above two equations

$$N = \left[\frac{nL^{\frac{3}{2}}}{3\sqrt{4\pi}}\right] f^{-3/2}$$

Since all the quantities in the bracket are constants so

$$\frac{logN}{logf} = -1.5$$

On the basis of this relation one can find the geometry of space.

4.3 Relativistic cosmology

4.3.1 FRW metric

In the last section it was discussed that metric of a homogeneous and isotropic universe can be written in the following form.

$$ds^2 = dt^2 - R_0^2 \left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right]$$

or

$$dl^2 = dt^2 R_0^2 [d\chi^2 + S_k^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)]$$

where

$$S_k(\chi) = \begin{pmatrix} \sin\chi \\ \sinh\chi \\ \chi \end{pmatrix} \text{for} \begin{pmatrix} k = 1 \\ k = -1 \\ k = 0 \end{pmatrix}$$

Now if we consider the Hubble expansion of the universe, which says that the distance between galaxies increases with time. In a homogeneous and isotropic universe this expansion can be parameterized by a single parameter $a(t)$. For the case of expansing universe R_0 no longer remain constant and it becomes $a(t)$ i.e., $R_0 = a(t)$. Now the metric will become

$$ds^2 = dt^2 - a^2(t) \left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right]$$

or

$$dl^2 = dt^2 - a^2(t)[d\chi^2 + S_k^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)] \tag{4.6}$$

It was found to be write this metric in terms of a new time η called the conformal time

$$d\eta = \frac{dt}{a(t)}$$

$$ds^2 = a^2(t) \left(d\eta^2 - \left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] \right)$$

and

$$dl^2 = a^2(t) \left(d\eta^2 - [d\chi^2 + S_k^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)] \right) \tag{4.7}$$

The above metric is called the Friedman-Robertson-Walker or FRW metric.

4.3.2 Cosmological redshift

In an expansing universe distances are represented by a parameter called redshift which is defined by the following way.

Let us consider a light source at (χ, θ, ϕ) which emits a light pulse at time η_e and this pulse is received by an observer at $(0, \theta, \phi)$ so

$$ds^2 = 0 = a^2(t)[d\eta^2 - d\chi^2] = 0$$

or

$$\chi = \pm\eta$$

from this equation it is clear that if two light pulses are emitted by the source at η_e and $\eta_e + d\eta_e$ and are received by the observer at η_o and $\eta_o + d\eta_o$ then

$$d\eta_e = d\eta_s$$

or

$$\frac{a(t_e)}{dt_e} = \frac{a(t_o)}{dt_o}$$

or

$$\frac{dt_o}{dt_e} = \frac{\nu_e}{\nu_o} = \frac{\lambda_o}{\lambda_e} = \frac{a(t_o)}{a(t_e)}$$

Since the universe expands so for any time $t_e < t_0$ we have $a(t_o) > a(t_e)$ or the wavelength of observed light is always greater than that of emitted light. This stretching of wavelengths due to cosmic expansion is quantified in terms of a parameter called the redshift 'z'.

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e}$$

$$1 + z = \frac{a(t_o)}{a(t_e)} \tag{4.8}$$

In general value of a at present time is taken '1' and for an object which is at redshift z the above expression becomes.

$$1 + z = \frac{1}{a} \tag{4.9}$$

From the redshift 'z' of an object one can find how far the object is or how fast the object is moving.

When astrophysical objects are not very far we can express cosmic scale factor by Taylor expansion.

$$a(t) = a(t_0) + (t - t_0)\dot{a}(t_0) + \frac{1}{2}(t - t_0)^2\ddot{a}(t_0) + ... \tag{4.10}$$

or

$$\frac{a(t)}{a(t_0)} = 1 + H_0(t - t_0) - \frac{1}{2}q_0H_0^2(t - t_0)^2 + ...$$

where

$$H_0 = (\frac{\dot{a}_0}{a_0}); q_0 = -(\frac{\ddot{a}_0 a_0}{\dot{a}_0^2}) \tag{4.11}$$

H_0 and q_0 are called the *Hubble parameter* and *deacceleration paramers* respectvely. From the above equations

$$\frac{1}{1+z} = 1 + H_0(t - t_0) - \frac{1}{2}q_0H_0^2(t - t_0)^2 + ...$$

or

$$z = H_0(t_0 - t) + \left(1 + \frac{q_0}{2}\right) H_0^2(t_0 - t)^2 + ... \tag{4.12}$$

This equation can solved for the look back time $(t_0 - t)$ also

$$t_0 - t = \frac{1}{H_0}\left[z - \left(1 + \frac{q_0}{2}\right) z^2 + ...\right]$$

4.3.3 Proper distance

Now let us consider an object at the point $(r_1\theta, \phi)$ emits a light pulse at time t which we receive at point $(0, \theta\phi)$ so from the FRW metric one can write

$$ds^2 = 0 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2}\right]$$

or

$$\int_0^{r_1} \frac{dr}{1-kr^2} = \int_0^t \frac{dt}{a(t)}$$

using the value of $\frac{1}{a(t)}$

$$\int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}} = r_1 + O(r^3) = \frac{1}{a_0}\int_{t_0}^t dt\frac{a_0}{a(t)} \tag{4.13}$$

but we know that

$$\frac{a_0}{a(t)} = 1 + H_0(t_0 - t) + ...$$

so for the case $O(r^3) \approx 0$

$$r_1 = \frac{1}{a_0}\left[(t_0 - t) + \frac{1}{2}H_0(t_0 - t)^2\right]$$

From the equation 4.12 we can put the compute the value of $t_0 - t$ i.e,

$$t_0 - t = \frac{1}{H_0}\left[z - (1 + \frac{q_0}{2})z^2\right]$$

so

$$r_1 = \frac{1}{H_0 a_0}\left[z - \frac{1}{2}(1 + q_0)z^2\right]$$

Here r_1 is called the *proper distance*. There are some other distances also which are very useful in observational cosmology.

4.3.4 Luminsoty distance

Let us consider a source of intrinsic luminosity L at a proper distance r_1 which emits a light pulse at frequence ν_e in time interval dt_e and this pulse is received by an observer position $r = 0$ at time t_0 in time interval dt_0.

Now it is straightforward to show that

$$\frac{dt_e}{dt_0} = \frac{a(t_e)}{a(t_o)}$$

If the energy of emitted and observed pulse are dE_e and dT_o respectvely then

$$\frac{dE_e}{dE_0} = \frac{a(t_0)}{a(t_e)}$$

Now from the above two equations

$$\frac{dE_e}{dt_e} = \left(\frac{a(t_0)}{a(t_e)}\right)^2 \frac{dE_o}{dt_o}$$

$$L_o = \left(\frac{a(t_e)}{a(t_o)}\right)^2 L$$

where $L = \frac{dE_e}{dt_e}$ is the absolute luminosity of the objects and $L_o = \frac{dE_o}{dt_o}$ is the apparent luminosity.

Now the apparent flux f will be

$$f = \frac{L_o}{4\pi r_1^2} = \left(\frac{a(t_e)}{a(t_o)}\right)^2 \frac{L}{4\pi r_1^2} = \frac{}{4\pi d_L^2}$$

where $r_1 = r_1(t_0)$ and

$$d_L = r_1 \left(\frac{a(t_o}{a(t_e)}\right) = r_1(t_0)(1+z) \tag{4.14}$$

Here d_L is called the *luminosity distance*. Since we know

$$r_1 = \frac{1}{H_0 a_0}\left[z - \frac{1}{2}(1+q_0)z^2\right]$$

so we can write the expression for luminosity distance in terms of redshift 'z' also.

$$d_L = \frac{1}{H_0}\left[z - \frac{1}{2}(1-q_0)z^2\right]$$

This is a very important relation, it says that how the luminosity distance change with redshift 'z' for a given values of H_0 and q_0. This relation can be used to find the value of H_0 or q_0 if any one of them is known or measuring the luminosity distances for some standard candles (objects known and constant intrinsic luminosity) at various 'z' one can measure the acceleration of the universe also. One can express the luminosity distance in terms of magnitude also.

$$m - M = 5 \log r_M pc + 25$$

in above expresions $r = d_l$ in terms of red shift z we can write

$$m - M = 5\log(\frac{d_l}{10pc})$$

$$= 5\log(\frac{cH_0^{-1}}{10pc}) + 5\text{log}_{10}(1+z) + 5\text{log}_{10}(\frac{r(z)}{cH_0^{-1}})$$

or

$$m - M = 5z + 5\text{log}_{10}(z)$$

distance modulus is defined as

$$dm = m - M = 5z$$

for small z

$$d_l(z) = \frac{2cH_0^{-1}}{\Omega_0^2}[(\Omega_0 - 2)[\sqrt{(1+\Omega_0 z)} - 1] + \Omega_0 z]$$

This formula is called the *matting formula* . In general our detectors are not equally sensitive for all wavelengths and the wavelength at which a source emit light and the wavelength at which the receiver receive that light are different due to expansion of the universe. In order to take this effect into account a terms called the *K correction* is added in the above expression.

Let intensity distribution function is $I(\ lambda)$ if wavelength of observed photon is λ_0 then the wavelength of emitted phton will be $\frac{\lambda_0}{(1+z)}$ .Thus an astronomer using a red filter may be actually receiving a photon that originated in the blue part of spectrum for a source at $z \approx 1$.

4.3.5 Angular dimeter distance

If today (t_0) physical distance of an object is ($r_1(t_0)$) then its physical distance at (t_1) will be ($r_1(t_1)$) where

$$\frac{r_1(t_0)}{r_1(t_1)} = \frac{a(t_0)}{a(t_1)}$$

and if its size is D and it subtends an angle θ at observers

$$\theta = \frac{D}{r_1(t_1)} = \frac{D}{r_1(t_0)\frac{a(t_1)}{a(t_o)}} = \frac{D}{d_A}$$

where

$$d_A = r_1(t_0)\frac{a(t_1)}{a(t_o)} = \frac{r_1(t_0)}{1+z} \tag{4.15}$$

From equation (4.14) and (4.15) we get

$$d_l = d_A(1+z)^2$$

4.4 Friedmann Models

FRW metric is given by

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{4.16}$$

One can compare it with the following metric can obtain the various components of $g_{\mu\nu}$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

Here we will use the following convention: μ or ν =0, 1, 2, 3 are for t, r, θ and ϕ respectively.

From the equation (4.16) we can compute the various components of Christoffel connections $\Gamma^\gamma_{\alpha\beta}$ which is defined as

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}g^{\gamma\lambda}\left(\partial_\beta g_{\alpha\lambda} + \partial_\alpha g_{\beta\lambda} - \partial_\lambda g_{\alpha\beta}\right)$$

Non zero components of $\Gamma^{\gamma}_{\alpha\beta}$ are as following

$$\Gamma^0_{ij} = -\frac{\dot{a}}{a} g_{ij}, \quad \Gamma^i_{j0} = \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta_i j \quad \text{and} \quad \Gamma^i_{jk} = \frac{1}{2} g^{il}\left(\partial_k g_{lj} + \partial_j g_{lk} - \partial_l g_{jk}\right) \tag{4.17}$$

where $i, j, k = 1, 2, 3$

We can also compute the components of the Ricci tensor $R_{\alpha\beta}$ which is defined as

$$R_{\alpha\beta} = \Gamma^{\sigma}_{\alpha\beta,\sigma} - \Gamma^{\sigma}_{\alpha\sigma,\beta} + \Gamma^{\sigma}_{\alpha\beta}\Gamma^{\rho}_{\sigma\rho} - \Gamma^{\sigma}_{\alpha\rho}\Gamma^{\rho}_{\beta\sigma}$$

The non zero components of this tensor are as follows:

$$R_{00} = -3\frac{\ddot{a}}{a} g_{00} \quad \text{and} \quad R_{ij} = R_{ji} = -\left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k}{a^2}\right] g_{ij} \tag{4.18}$$

One can compute the Ricci scalar 'R' also

$$R = R_{\alpha\beta} g^{\alpha\beta} = -\frac{6}{a^2}[a\ddot{a} + \dot{a}^2 + k] \tag{4.19}$$

4.4.1 Einstein's equations

Einstein's equations are defined as follows:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta} \;\text{ or }\; G_{\alpha\beta} = 8\pi G T_{\alpha\beta} \;\text{ where }\; G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \tag{4.20}$$

Here $T_{\alpha\beta}$ is the Energy momentum tensor. We can solve this set of equations for a simple case of perfect fluid for which $T_{\alpha\beta}$ is defined as following:

$$T_{\alpha\beta} = (p + \rho) U_\alpha U_\beta - p g_{\alpha\beta} \tag{4.21}$$

Here this tenor is written in a frame in which the universe is not expanding or the spatial components of the four velocity U_α are zero i.e., $U_\alpha = (1, 0, 0, 0)$ in a system of units for which 'c=1'. From the equation (4.21) it is easy to find that

$$T_{00} = \rho g_{00} \quad \text{and} \quad T_{ij} = -p g_{ij} \tag{4.22}$$

Now from the equations (4.18), (4.19) and (4.16) we can compute the various components of $G_{\alpha\beta}$ which is defined in equation (4.20):

$$G_{00} = 3\left[\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right] g_{00} \;\text{ and }\; G_{ij} = \left[2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right] g_{ij} \tag{4.23}$$

Now the zeroth component of Einstein's equation i.e., $G_{00} = 8\pi G T_{00}$ gives

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G\rho}{3} \tag{4.24}$$

The spatial components $G_{ij} = 8\pi G T_{ij}$ give

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G p \tag{4.25}$$

using equation (4.24) this can be written as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{4.26}$$

Equation (4.24) & (4.40) are called the Friedmann's equations.

4.4.2 Einstein's static universe

Let us see how the static solutions i.e., $\ddot{a} = \dot{a} = 0$ of equation (4.24) & (4.40) look for a pressure-less dust i.e., $p = 0$

Equation (4.24) becomes

$$\frac{k}{a^2} = \frac{8\pi G\rho}{3}$$

and equation (4.40) becomes

$$0 = -\frac{4\pi G}{3}(\rho + 3p)$$

One can see that apart from the case of empty universe ($p = \rho = 0$) these equations require either negative p or ρ for their solutions.

In order to find a static solution of Einstein introduced a term named cosmological constant or Λ in his equations

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi G T_{\alpha\beta} + \Lambda g_{\alpha\beta} \tag{4.27}$$

Now it is easy to see that in this the Friedmann's equations become

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} \tag{4.28}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \tag{4.29}$$

With this modification we have the following equations for the static case

$$\frac{k}{a^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3}$$

$$0 = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}$$

Now it is easy to see that there exist the following solution for these equations for the case of pressure-less dust

$$\rho = \frac{\Lambda}{4\pi G} \quad \text{and} \quad \frac{k}{a^2} = \Lambda$$

If we consider the case 'k=1' then we get

$$a = \frac{1}{\sqrt{\Lambda}}$$

Since we know that the volume of three sphere S^3 of radius R embedded in four dimensional Eucledian space is $2\pi^2 R^3$ so one can compute the volume of the universe in the above case also.

$$V = 2\pi^2 a^3 = 2\pi^2 \Lambda^{-\frac{3}{2}} \tag{4.30}$$

From the above discussion it is also clear that we can associate an energy density with the cosmological constant

$$\rho_\Lambda = \frac{\Lambda}{4\pi G} \tag{4.31}$$

The main feature of this model which is called the *Einstein's static model* is that in this case the universe does not expands i.e.,$a(t) = constant$, however, it does contain matter.

4.4.3 de-Sitter's expanding universe

There exist one such solution also in which the universe does not contain matter but it expands; this solution is called the *de-Sitter solution.* In this case equations (4.37) and (4.36) become

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{\Lambda}{3} \tag{4.32}$$

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3} \tag{4.33}$$

We can differentiate equation (4.32) and put the value of $\ddot{a}$ from the (4.33) and obtain

$$\dot{a}^2 = \Lambda\frac{a^2}{3}$$

This equations has the following solution

$$a(t) = a(t_0)e^{\sqrt{\frac{\Lambda}{3}}t} = a(t_0)e^{H_0 t} \quad \text{where} \quad H_0 = \frac{\dot{a}(t_0)}{a(t_0)} = \sqrt{\frac{\Lambda}{3}} \tag{4.34}$$

In this case the universe expands exponentially and its rate of expansion remains constant.

Friedmann's equations are as follows

$$(\frac{\dot{a}}{a})^2 + \frac{k}{a^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} \tag{4.35}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + p) + \frac{\Lambda}{3} \tag{4.36}$$

4.4.4 Density parameter & deacceleration parameter

One can find the curvature 'k' of the spatial part of the universe from the following Friedmann's equation.

$$(\frac{\dot{a}}{a})^2 + \frac{k}{a^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} \tag{4.37}$$

Since we know $\frac{\dot{a}}{a} = H(t) = H$ so this equation also can be written as

$$1 + \frac{k}{H^2a^2} = \frac{8\pi G\rho}{3H^2} + \frac{\Lambda}{3H^2}$$

or

$$1 + \frac{k}{H^2a^2} = \frac{8\pi G(\rho + \rho_\Lambda)}{3H^2} \quad \text{where} \quad \rho_\Lambda = \frac{\Lambda H^2}{8\pi G}$$

Defining

$$\rho_t = \rho + \rho_\Lambda \quad \text{and} \quad \rho_c = \frac{3H^2}{8\pi G} \tag{4.38}$$

The above equation can be written as

$$1 + \frac{k}{H^2a^2} = \frac{\rho_t}{\rho_c} = \Omega$$

Where $\Omega = \frac{\rho}{\rho_c}$ is called the *density parameter*. Now in terms of Ω we can find the value of 'k'

$$k = H^2 a^2 (\Omega - 1)$$

This expression clearly shows that

$$k = \begin{cases} 0, & \text{if} \quad \Omega = 1 \\ +1, & \text{if} \quad \Omega > 1 \\ -1, & \text{if} \quad \Omega < 1 \end{cases} \tag{4.39}$$

Now the second Friedmann's equation is

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{4.40}$$

or

$$\left(\frac{-a\ddot{a}}{\dot{a}^2}\right)\frac{\dot{a}^2}{a^2} = \frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}$$

Let us put use the following variable

$$q = \left(\frac{-a\ddot{a}}{\dot{a}^2}\right) \quad \text{and} \quad H = \frac{\dot{a}}{a}$$

so the equation becomes

$$q = \frac{\Omega}{2} + \frac{\Lambda}{3H^2}$$

from the first Friedmann's equation we find

$$1 + \frac{k}{a^2 H^2} = \Omega + \frac{\Lambda}{3H^2}$$

Eliminating Λ from these two equations

$$k = a^2 H^2 (\frac{\Omega}{2} + q - 1)$$

If there is no cosmological constant i.e., $\Lambda = 0$ the we get

$$k = a^2 H^2 (\Omega - 1) = a^2 H^2 (2q - 1)$$

writing this equation at present time t_0

$$k = a_0^2 H_0^2 (2q_0 - 1)$$

4.4.5 Equation of state

There are three variables a, ρ and p and we have only two equations. In order to make this set complete we need one more equation i.e., equation of state

$$p = w\rho \tag{4.41}$$

Moreover, using entropy conservation equation i.e., $dU + pdV = 0$ one can find a very useful expression for pressure ρ in terms of a

$$d(\rho a^3) = -pd(a^3)$$

or

$$3a^2(\rho + p) = -a^3 d\rho \quad \text{or} \quad 3(\rho + p) = -a d\rho$$

now using equation (4.41) we get

$$\frac{d\rho}{\rho} = -3(1+w)\frac{da}{a}$$

which has the following solution

$$\rho(a) = \frac{1}{a^{3(1+w)}} \tag{4.42}$$

For some special cases form of this equation is as follows:

$$\rho(a) \propto \begin{cases} \frac{1}{a^3}, & \text{for Non-relativistic matter, because} \quad w = 0 \\ \frac{1}{a^4}, & \text{for relativistic matter, because} \quad w = \frac{1}{3} \\ \text{constant} & \text{for } \Lambda\text{, because} \quad w = -1 \end{cases} \tag{4.43}$$

Friedmamm's equations:

Now using the equation of state one can write Friedmann's equations for a universe which contains relativistic matter, relativistic matter and the cosmological constant in the following form:

$$\frac{\dot{a}^2}{a} + \frac{k}{a^2} = \frac{8\pi G}{3}\left[\rho_0^{NR}\left(\frac{a_0}{a}\right)^3 + \rho_0^R\left(\frac{a_0}{a}\right)^4 + \rho_0^\Lambda\right]$$

Here subscript represent the values at the present time. Note that here

$$\rho^\Lambda = \frac{\Lambda H_0^2}{8\pi G}$$

It is important to note that in this formalism a_0 has the unit of length, so the units of Λ are $(length)^2$. Sometime $\frac{1}{8\pi G}$ is written in terms of Planck's unit also i.e., $\frac{1}{8\pi G} = M_P^2$ where M_P is the planck mass.

We can put the value k and use density parameter

$$\frac{\dot{a}^2}{a^2} = H^2(a) = H_0^2\left[\Omega_0^{NR}\left(\frac{a_0}{a}\right)^3 + \Omega_0^R\left(\frac{a_0}{a}\right)^4 + \Omega_0^\Lambda + (1-\Omega_0)\left(\frac{a_0}{a}\right)^2\right] \tag{4.44}$$

Here

$$\Omega_0^{NR} = \frac{8\pi G\rho_0^{NR}}{3H_0^3}; \Omega_0^R = \frac{8\pi G\rho_0^R}{3H_0^3}; \Omega_0^\Lambda = \frac{8\pi G\rho_0^\Lambda}{3H_0^3}$$

Since in place of a, observable quantity is z so the above can be written in the form of z

$$1 + z = \frac{a_0}{a} \quad \text{or} \quad \frac{a}{a_0} = \frac{1}{1+z}$$

It gives

$$H^2(z) = H_0^2\left[\Omega_0^{NR}(1+z)^3 + \Omega_0^R(1+z)^4 + \Omega_0^\Lambda + (1-\Omega_0)(1+z)^2\right] \tag{4.45}$$

4.4.6 Various cosmic epochs

The above equation clearly shows that density contributed by cosmological constant and non-relativistic matter increse with slower rate as comparison to relativtic matter when we go back in time i.e., $a \longrightarrow 0$. This allow us to make the assumption that the earliest epoch of the universe was dominated by relativistic matter. Moreover, we can find an time at which density of non-relativistic matter and relativistic matter were equal. This epoch is called radiation-matter equality a_{equ} or z_{equ}.

$$\left(\frac{a_0}{a_{equ}}\right) = 1 + z_{equ} = \frac{\Omega_0^{NR}}{\Omega_0^R}$$

Use $\Omega_0^{NR} \approx 0.3$ and $\Omega_0^R \approx 2 \times 10^{-5}$ we get $z_{equi} \approx 15000$. By the same one can find the equalities for non-relativistic-curvature and curvature-cosmological constant also.

One can put all solutions of equation (4.46) and (4.45) into two classes: flat models and curved models, i.e., $k = 0$ and $k \neq 0$.

4.5 Special case $k = 0$: Flat universe

In these models $k = 0$ or $\Omega_0 = 1$. So the above equation (4.46) reduced in the following form:

$$\frac{\dot{a}^2}{a^2} = H(a) = H_0^2 \left[\Omega_0^{NR}\left(\frac{a_0}{a}\right)^3 + \Omega_0^R\left(\frac{a_0}{a}\right)^4 + \Omega_0^\Lambda\right] \tag{4.46}$$

In order to solve these equation we need to know the values of four parameters $\Omega_0^{NR}, \Omega_0^R, \Omega_0^\Lambda$ and H_0. Since we have one extra condition i.e., $\Omega_0^{NR} + \Omega_0^R + \Omega_0^\Lambda = 1$ so finally we need three parameters which must be measured observationally. The exact solution of the above equation will depend on the case, however, we can solve the above equation for some simple cases.

4.5.1 Non-relativistic matter dominated

This case is characterized by $\Omega_0^{NR} = 1, \Omega_0^R = 0, \Omega_0^\Lambda = 0$

So we get

$$\frac{\dot{a}}{a} = H_0\left(\frac{a_0}{a}\right)^{\frac{3}{2}}$$

or

$$\int a^{\frac{1}{2}} da = H_0 a_0^{\frac{3}{2}} \quad \text{or} \quad a^{\frac{3}{2}} = \frac{3}{2} H_0 a_0^{\frac{3}{2}} t$$

or

$$a(t) = a_0\left(\frac{t}{t_0}\right)^{\frac{3}{2}} \quad \text{where} \quad t_0 = \frac{2}{3H_0} \tag{4.47}$$

This solution is called the **Einstein-de Sitter solution** . In this case the universe starts with a big bang $(t = 0, a = 0)$ and expand indefinitely.

4.5.2 Relativistic matter dominated

This case is characterized by $\Omega_0^{NR} = 0, \Omega_0^R = 1, \Omega_0^\Lambda = 0$

So we get

$$\frac{\dot{a}}{a} = H_0 \left(\frac{a_0}{a}\right)^2$$

or

$$\int a da = H_0 a_0^2 \text{ or } a^2 = 2H_0 a_0^2 t$$

or

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^2 \quad \text{where} \quad t_0 = \frac{1}{2H_0} \tag{4.48}$$

4.5.3 Cosmological constant dominated

This case is characterized by $\Omega_0^{NR} = 0, \Omega_0^R = 0, \Omega_0^\Lambda = 1$

It gives

$$\frac{\dot{a}}{a} = H_0$$

or

$$\int \frac{da}{a} = H_0 a_0 t \text{ or } \ log(a/a_0) = H_0 t$$

or

$$a(t) = a_0 e^{\frac{t}{t_0}} \text{ where } \quad t_0 = \frac{1}{H_0}$$

From the above discussion this is clear that the rate of expansion of the universe in maximum in cosmological constant dominated phase and it is minimum in matter dominated phase.

4.5.4 ΛCDM model

In this model it is considered that the universe is dominated by two species: cosmological constant and non-relativistic cold dark matter. Before going to put the exact values of density parameters we can try to integrate the equation for a general case.

$$\frac{\dot{a}}{a} = H_0 \left[\Omega_0^{NR} \left(\frac{a_0}{a}\right)^3 + \Omega_0^\Lambda\right]^{\frac{1}{2}}$$

It gives

$$t = \frac{1}{H_0 \sqrt{\Omega_0^{NR}}} \int \frac{\sqrt{y} dy}{\sqrt{1 + \frac{\Omega_0^\Lambda}{\Omega_0^{NR}} y^3}}$$

Now we can integrate it by the following substitution

$$\frac{\Omega_0^\Lambda}{\Omega_0^{NR}} y^3 = \sinh^2 \theta$$

This gives

$$t = \frac{2}{3H_0}\frac{1}{\sqrt{\Omega_0^\Lambda}}\sinh^{-1}\sqrt{\left[\frac{\Omega_0^\Lambda}{\Omega_0^{NR}}\left(\frac{a}{a_0}\right)^3\right]} = \frac{2}{3H_0}\frac{1}{\sqrt{\Omega_0^\Lambda}}\sinh^{-1}\sqrt{\left[\frac{\Omega_0^\Lambda}{\Omega_0^{NR}}(1+z)^3\right]}$$

One can write this in other form also

$$a(t) = a_0\left[\frac{\Omega_0^{NR}}{\Omega_0^\Lambda}\right]^{\frac{1}{3}}\sinh^{\frac{2}{3}}\left[\frac{3H_0t}{2}\sqrt{\Omega_0^\Lambda}\right]$$

We know that

$$\sinh\theta = \frac{e^\theta - e^{-\theta}}{2}$$

This gives $\sinh\theta = \theta$ for $\theta << 1$ and $\sinh\theta = e^\theta$ for $\theta >> 1$ which means for $\Omega_0^\Lambda << 1$ we get $a \propto t^{\frac{2}{3}}$ and for $\Omega_0^\Lambda >> 1$ we get $a \propto e^{H_0t}$. These results we already know.

4.6 Special case $k \pm 1$: Curved universe

Let us consider a case in which the dominant component of the universe is non relativistic matter. In this case we have to solve the following equation:

$$\frac{\dot{a}^2}{a^2} = H_0^2\left[\Omega_0\left(\frac{a_0}{a}\right)^3 + (1-\Omega_0)\left(\frac{a_0}{a}\right)^2\right] \tag{4.49}$$

This gives

$$t = \frac{1}{H_0\sqrt{\Omega_0}}\frac{1}{a_0^{\frac{3}{2}}}\int\frac{a^{\frac{1}{2}}da}{\left[1+\frac{(1-\Omega_0)}{\Omega_0}\left(\frac{a}{a_0}\right)\right]^{\frac{1}{2}}}$$

Now this equation has different solutions for the following two different cases:

4.6.1 Closed Model: $k = 1$ or $\Omega_0 > 1$ case

In order to solve the above integral we make the following substitution:

$$\frac{a}{a_0} = \frac{\Omega_0}{(\Omega_0 - 1)}\sin^2\frac{\theta}{2} = \frac{\Omega_0}{(\Omega_0 - 1)}\frac{1}{2}(1-\cos\theta)$$

This expression can be written in terms of redshift also.

$$\frac{1}{(1+z)} = \frac{\Omega_0}{(\Omega_0 - 1)}\frac{1}{2}(1-\cos\theta)$$

or

$$\theta = \cos^{-1}\frac{\Omega_0 z - \Omega_0 + 2}{(1+z)\Omega_0} \tag{4.50}$$

Note that for a closed universe i.e., $k = +1$

$$a_0 = \frac{1}{H_0\sqrt{\Omega_0 - 1}}$$

This means that

$$a = \frac{1}{H_0}\left[\frac{\Omega_0}{(\Omega_0 - 1)^{\frac{3}{2}}}\right]\frac{1}{2}(1 - \cos\theta)$$

making the above substitution we get the the following solution for t:

$$t = \frac{1}{H_0}\left[\frac{\Omega_0}{(\Omega_0 - 1)^{\frac{3}{2}}}\right]\frac{1}{2}(\theta - \sin\theta)$$

Substituting the value of θ and $\sin\theta$

$$t = \frac{1}{2H_0}\left(\frac{\Omega_0}{(\Omega_0 - 1)^{\frac{3}{2}}}\right)\left[\cos^{-1}\left(\frac{\Omega_0 z - \Omega_0 + 2}{(1+z)\Omega_0}\right) - \frac{2(\Omega_0 - 1)^{\frac{1}{2}}(\Omega_0 z + 1)^{\frac{1}{2}}}{\Omega_0(1+z)}\right] \tag{4.51}$$

These solutions can be written in terms of the following form:

$$a(\theta) = A(1 - \cos\theta) \text{ and } t(\theta) = B(\theta - sin\theta)$$

We can see that a starts from minimum i.e., $a = 0$ at $\theta = 0$ and goes maximum i.e., $a_{max} = 2A$ at $\theta = \pi, t_{max} = B\pi$ and then finally it collapse to a singular point. One can write the above solution in a convinent form.

$$\frac{a}{a_{max}} = \frac{1}{2}(1 - \cos\theta) \text{ and } \frac{t}{t_{max}} = \frac{1}{\pi}(\theta - \sin\theta) \tag{4.52}$$

4.6.2 Open Model: $k = -1$ or $\Omega_0 < 1$ case

In this case we need the following substitution:

$$\frac{a}{a_0} = \frac{\Omega_0}{(1 - \Omega_0)}\sinh^2\frac{\theta}{2} = \frac{\Omega_0}{(1 - \Omega_0)}\frac{1}{2}(\cosh\theta - 1)$$

or

$$a(\theta) = \frac{1}{H_0}\left[\frac{\Omega_0}{(1 - \Omega_0)^{\frac{3}{2}}}\right]\frac{1}{2}(\cosh\theta - 1)$$

This gives the following solution:

$$t = \frac{1}{H_0}\left[\frac{\Omega_0}{(1 - \Omega_0)^{\frac{3}{2}}}\right]\frac{1}{2}(\sinh\theta - \theta)$$

Again putting back the value of θ we get

$$t = \frac{1}{2H_0}\left(\frac{\Omega_0}{(1 - \Omega_0)^{\frac{3}{2}}}\right)\left[-\cosh^{-1}\left(\frac{\Omega_0 z - \Omega_0 + 2}{(1+z)\Omega_0}\right) + \frac{2(1 - \Omega_0)^{\frac{1}{2}}(\Omega_0 z + 1)^{\frac{1}{2}}}{\Omega_0(1+z)}\right] \tag{4.53}$$

These solutions can be written in terms of the following form:

$$a(\theta) = A(\cosh\theta - 1) \text{ and } t(\theta) = B(\sinh\theta - \theta)$$

In this case the universe expands idefnitely. can plot a for various values of t (see Figure 4.1).

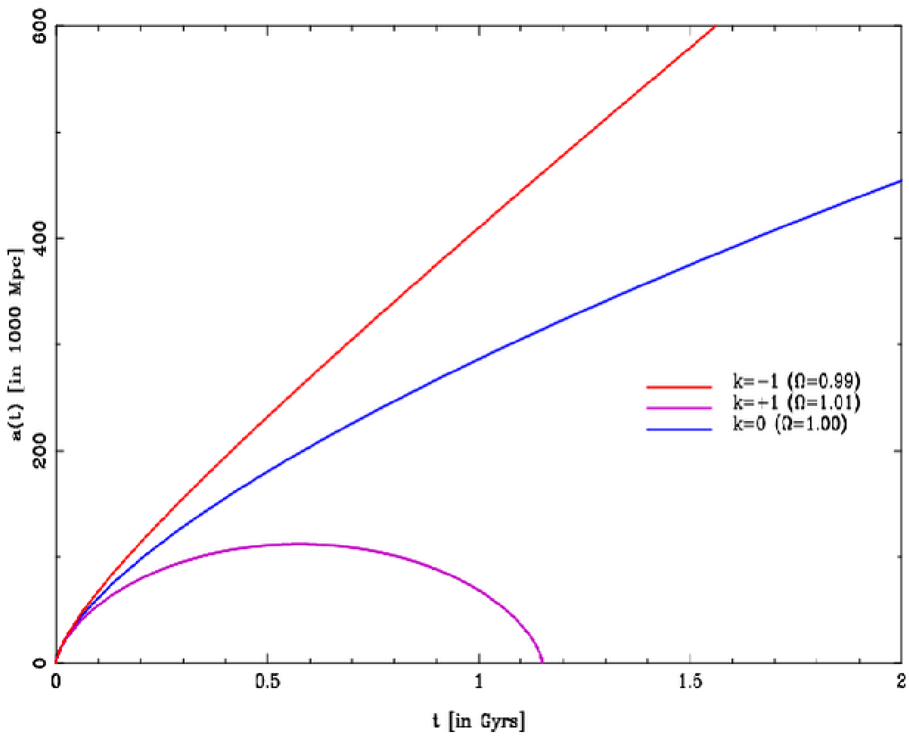


Figure 4.1: Expansion of the universe in three different case. Here hubble constant at present is taken as 100 km/sec/Mpc.

4.7 Size and age of of the universe

4.7.1 Hubble size

Expansion rate H of the universe is governed by the following equation:

$$H(a) = \frac{\dot{a}}{a} = H_0 \left[\Omega_0^{NR} \left(\frac{a_0}{a} \right)^3 + \Omega_0^R \left(\frac{a_0}{a} \right)^4 + \Omega_0^\Lambda + (1 - \Omega_0) \left(\frac{a_0}{a} \right)^2 \right]^{\frac{1}{2}}$$

or

$$H(z) = H_0 \left[\Omega_0^{NR} (1+z)^3 + \Omega_0^R (1+z)^4 + \Omega_0^\Lambda + (1 - \Omega_0)(1+z)^2 \right]^{\frac{1}{2}}$$

H has unit of time so we can find a distance called the *Hubble's length* (d_H) , that will be traveled by light in this time. Physically this means that beyond this distance galaxies travel with superluminal speeds.

$$d_H = \frac{c}{H}$$

We can compute its value for the following simple cases:

1. Non-relativistic & Flat: $\Omega_{NR} = 1$

   This gives

$$d_H = \frac{c}{H_0} \left(\frac{a}{a_0} \right)^{\frac{3}{2}} = \frac{c}{H_0} (1+z)^{-\frac{3}{2}}$$

2. Relativistic & Flat: $\Omega_R = 1$

   For this case

$$d_H = \frac{c}{H_0}\left(\frac{a}{a_0}\right)^2 = \frac{c}{H_0}(1+z)^{-2}$$

3. Λ & Flat: $\Omega_\Lambda = 1$

$$d_H = \frac{c}{H_0}$$

   This shows that in this case the size of hubble length does not change.

   Note that only those physical process can influence each other which are within a distance less than the Hubble length.

4.7.2 Particle horizon

One of the important consequences of the big-bang theory is that there exist a particle horizon, i.e., there is a maximum distance at any time that light can travel since the initial singularity ($a = 0$ or $z = \infty$). FRW metric is given by

$$ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta + \sin^2\theta d\phi^2)\right]$$

Let us consider the propagation of a light ray along radial direction i.e., constant θ and ϕ.

$$0 = c^2dt^2 - a^2(t)\frac{dr^2}{1-kr^2}$$

So the comoving physical distance $\chi_k(r)$ can be computed as follows:

$$d\chi_k(r) = \frac{dr}{\sqrt{1-kr^2}} = \frac{cdt}{a(t)} = \frac{cda}{a\frac{da}{dt}} = \frac{cda}{a^2H(a)}$$

In order to compute the size of particle horizon at any time, which is defined by $d_P(a) = a(t)\chi_k(r(a))$, we need to know the full form of $H(a)$, however, if we assume that the solution can be written in the following form:

$$a(t) \propto t^n$$

then one can compute $d_P(a)$ easily

$$d_P(t) = a(t)\int_0^t \frac{cdt}{a(t)} = t^n\frac{t^{-n+1}}{-n+1} = \frac{ct}{1-n}$$

Note that it increases linearly with time, however, the coefficient deepens of the exact form of $a(t)$. We can compute d_P for the following two simple cases.

1. Matter dominated flat universe:

   For this case $a \propto t^{\frac{2}{3}}$ so $n = \frac{2}{3}$. It gives

$$d_P(t) = 3ct$$

2. Radiation dominated flat universe:-

$$a(t) \propto t^{\frac{1}{2}} \quad \text{or} \quad n = \frac{1}{2}$$

   It gives

$$d_P(t) = 2t$$

3. Cosmological constant dominated universe :-

$$a(t) \propto e^{H_0 t}$$

$$d_P(t) = \frac{1}{H_0}(e^{H_0 t} - 1)$$

For a generic case size of particle horizon can be computed as following.

$$d_P(t) = a(t) \int_0^t \frac{cdt}{a} = a(t) \int_0^t \frac{cda}{a^2 H(a)}$$

Where

$$H(a) = H_0 \left[\sum_{i=1,3} \Omega_0^i \left(\frac{a_0}{a} \right)^{n_i} - (1 - \Omega_0) \left(\frac{a_0}{a} \right)^2 \right]^{\frac{1}{2}}$$

Here $i = 1, 2$ and 3 for non-relativistic matter, relativistic matter and cosmological constant respectvely. It is clear that $n_1 = 3, n_2 = 4$ and $n_3 = 0$.

Dynamics of the universe:

Geometrical structure of our space-time can be given by a homogeneous and isotropic metric called the *Friedmann line element.*

$$ds^2 = c^2 dt^2 - a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right) \tag{4.54}$$

Here r is measured in terms of the *curvature length* of the spatial part, k is called the spatial curvature and takes three values $\pm 1, 0$ for spatially closed, open and flat universe. In a *simply connoted* case spatial volume of the universe is finite for $k = +1$ and infinite for $k = -1$, and 0 case.

There are two parameters a and k in this metric and in order to fix these Einstein's equations are used which take a very simple form for a homogeneous and isotropic universe:

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} \tag{4.55}$$

$$2\frac{\ddot{a}}{a}+\left(\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right)=-8\pi Gp+\Lambda \tag{4.56}$$

From equation (5.48)

$$k=a_0^2H_0^2(\Omega_0-1) \text{ where } \Omega_0=\sum_{i=NR,R,\Lambda}\Omega_{i,0} \tag{4.57}$$

Where $\rho_\Lambda=\frac{\Lambda}{8\pi G}$, $H=\frac{\dot{a}}{a}$, $\Omega=\frac{\rho}{\rho_c}$, $\rho_c=\frac{3H^2}{8\pi G}$ and subscript "0" reprsents the present values i.e., at $t=t_0$.

Using the fact $\rho_{NR,R,\Lambda}\propto a^{-3,-4,0}$, and equation (4.57) we can write equation (5.48) in the following convinent form:

$$H^2(a)=\frac{\dot{a}^2}{a^2}=H_0^2\left[\Omega_{NR,0}\left(\frac{a}{a_0}\right)^3+\Omega_{R,0}\left(\frac{a}{a_0}\right)^4+(1-\Omega_0)\left(\frac{a}{a_0}\right)^2+\Omega_\Lambda\right]^{\frac{1}{2}} \tag{4.58}$$

4.8 Age of the universe

From the above equation we can write:

$$t_0=\frac{1}{H_0}\int_0^1\frac{xdx}{\sqrt{\Omega_{R,0}+\Omega_{NR,0}x+\Omega_{k,0}x^2+\Omega_\Lambda x^4}} \tag{4.59}$$

Note that here $\Omega_{k,0}=1-\Omega_0$.

Age of the universe t_0 is defined by the time which the universe takes to expand from the big bang $a=0$ to present $a=a_0$.

In order to find the age of the universe in a cosmological model we need to know the values of four independent parameters $\Omega_{R,0},\Omega_{NR,0},\Omega_{k,0}$ and Ω_Λ, and of course value of Hubble constant H_0. In a general case the above integral cannot be solved analytically. However, for some of the following cases it can be solved analytically.

1. $\Omega_{NR,0}=1,\Omega_{R,0}=\Omega_{k,0}=\Omega_\Lambda=0$ (Einstein-de Sitter model):

$$t_0=\frac{2}{3H_0} \text{ and } a(t)=a(t_0)\left(\frac{t}{t_0}\right)^{3/2} \tag{4.60}$$

2. $\Omega_{R,0}=1,\Omega_{NR,0}=\Omega_{k,0}=\Omega_\Lambda=0$:

$$t_0=\frac{1}{2H_0} \text{ and } a(t)=a(t_0)\left(\frac{t}{t_0}\right)^2 \tag{4.61}$$

3. $\Omega_{k,0}=1,\Omega_{NR,0}=\Omega_{R,0}=\Omega_\Lambda=0$:

$$t_0=\frac{1}{H_0} \text{ and } a(t)=a(t_0)\left(\frac{t}{t_0}\right) \tag{4.62}$$

4. $\Omega_\Lambda = 1, \Omega_{NR,0} = \Omega_{R,0} = \Omega_{k,0} = 0$:

$$t_0 =? \quad \text{and} \quad a(t) = a(t_0)e^{t/t_0} \tag{4.63}$$

Some of the other cases which are a bit difficult to calculate are as follows:

5. $\Omega_{k,0} \geq 0, \Omega_{NR,0} \neq 0, \Omega_{R,0} = \Omega_\Lambda = 0$:

$$t = \frac{1}{H_0}\int_0^{\frac{a}{a_0}} \frac{dx}{\sqrt{\Omega_{k,0} + \Omega_{NR,0}x^{-1}}} = \frac{1}{H_0}\frac{1}{\sqrt{\Omega_{NR,0}}}\int_0^{\frac{a}{a_0}} \frac{x^{\frac{1}{2}}dx}{\sqrt{1 + x\frac{\Omega_{NR,0}}{\Omega_{k,0}}}} \tag{4.64}$$

This equation can be solved in the following parametric form:

$$a = \frac{a_0}{2}\frac{\Omega_{NR,0}}{\Omega_{k,0}}(\cosh\theta - 1) \tag{4.65}$$

$$t = \frac{1}{2H_0}\frac{\Omega_{NR,0}}{\Omega_{k,0}^{\frac{3}{2}}}(\sinh\theta - \theta) \tag{4.66}$$

and

$$t = \frac{1}{2H_0}\frac{\Omega_{NR,0}}{\Omega_{k,0}^{\frac{3}{2}}}\left[2\left(\frac{\Omega_{k,0}}{\Omega_{NR,0}}\right)^{\frac{1}{2}}\left(\frac{a}{a_0}\right)^{\frac{1}{2}}\left(1 + \left(\frac{\Omega_{k,0}}{\Omega_{NR,0}}\right)\left(\frac{a}{a_0}\right)\right)^{\frac{1}{2}} - \cosh^{-1}\left(1 + 2\left(\frac{\Omega_{k,0}}{\Omega_{NR,0}}\right)\left(\frac{a}{a_0}\right)\right)\right] \tag{4.67}$$

From this we can write the expression for the age of the universe:

$$t_0 = \frac{1}{2H_0}\frac{\Omega_{NR,0}}{\Omega_{k,0}^{\frac{3}{2}}}\left[2\left(\frac{\Omega_{k,0}}{\Omega_{NR,0}}\right)^{\frac{1}{2}}\left(1 + \left(\frac{\Omega_{k,0}}{\Omega_{NR,0}}\right)\right)^{\frac{1}{2}} - \cosh^{-1}\left(1 + 2\left(\frac{\Omega_{k,0}}{\Omega_{NR,0}}\right)\right)\right] \tag{4.68}$$

Now since in this case $\Omega_0 = \Omega_{NR,0}$ so we can replace $\Omega_{k,0} = 1 - \Omega_0 = 1 - \Omega_{NR,0}$ and this gives us:

$$t = \frac{\Omega_0}{2H_0(1-\Omega_0)^{\frac{3}{2}}}\left[2\left(\frac{(1-\Omega_0)}{\Omega_0}\right)^{\frac{1}{2}}\left(\frac{a}{a_0}\right)^{\frac{1}{2}}\left(1 + \left(\frac{(1-\Omega_0)}{\Omega_0}\right)\left(\frac{a}{a_0}\right)\right)^{\frac{1}{2}} - \cosh^{-1}\left(1 + 2\left(\frac{(1-\Omega_0)}{\Omega_0}\right)\left(\frac{a}{a_0}\right)\right)\right]$$

and

$$t_0 = \frac{\Omega_0}{2H_0(1-\Omega_0)^{\frac{3}{2}}}\left[2\left(\frac{(1-\Omega_0)^{\frac{1}{2}}}{\Omega_0}\right) - \cosh^{-1}\left(\frac{2}{\Omega_0} - 1\right)\right]$$

6. $\Omega_{k,0} \leq 0, \Omega_{NR,0} \neq 0, \Omega_{R,0} = \Omega_\Lambda = 0$:

$$t = \frac{1}{H_0}\int_0^{\frac{a}{a_0}} \frac{dx}{\sqrt{\Omega_{k,0} + \Omega_{NR,0}x^{-1}}} = \frac{1}{H_0}\frac{1}{\sqrt{\Omega_{NR,0}}}\int_0^{\frac{a}{a_0}} \frac{x^{\frac{1}{2}}dx}{\sqrt{1 + x\frac{\Omega_{NR,0}}{\Omega_{k,0}}}} \tag{4.69}$$

This equation can be solved in the following parametric form:

$$a = \frac{a_0}{2}\frac{\Omega_{NR,0}}{\Omega_{k,0}}(1-\cos\theta) \tag{4.70}$$

$$t = \frac{1}{2H_0}\frac{\Omega_{NR,0}}{\Omega_{k,0}^{\frac{3}{2}}}(\theta-\sin\theta) \tag{4.71}$$

and

Note that in this case a is a periodic function and its values is maximum for $a=\pi$

$$a_{max} = a_0\frac{\Omega_{NR,0}}{\Omega_{k,0}} \quad \text{and} \quad t_{max} = \frac{1}{2H_0}\frac{\Omega_{NR,0}}{\Omega_{k,0}^{\frac{3}{2}}}\pi$$

So in terms of these variables we can write:

$$a = a_{max}(1-\cos\theta) \quad \text{and} \quad t = \frac{t_{max}}{\pi}(\theta-\sin\theta)$$

$$t = \frac{\Omega_0}{2H_0(1-\Omega_0)^{\frac{3}{2}}}\left[-2\left(\frac{(\Omega_0-1)}{\Omega_0}\right)^{\frac{1}{2}}\left(\frac{a}{a_0}\right)^{\frac{1}{2}}\left(1+\left(\frac{(\Omega_0-1)}{\Omega_0}\right)\left(\frac{a}{a_0}\right)\right)^{\frac{1}{2}}+\cos^{-1}\left(1+2\left(\frac{(\Omega_0-1)}{\Omega_0}\right)\left(\frac{a}{a_0}\right)\right)\right]$$

and

$$t_0 = \frac{\Omega_0}{2H_0(\Omega_0-1)^{\frac{3}{2}}}\left[-2\left(\frac{(\Omega_0-1)^{\frac{1}{2}}}{\Omega_0}\right)+\cos^{-1}\left(\frac{2}{\Omega_0}-1\right)\right]$$

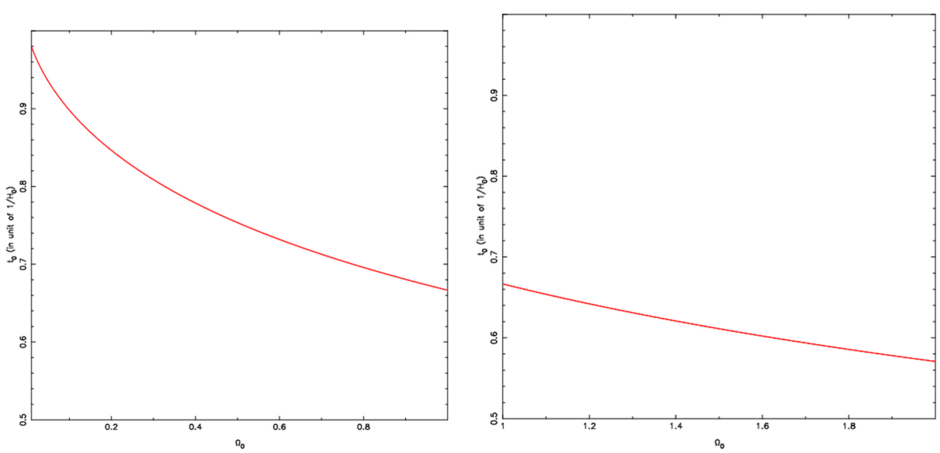


Figure 4.2: Age of the universe for an open (left) and closed (right) model of the universe

7. $\Omega_{NR,0}\neq 0, \Omega_\Lambda \neq 0, \Omega_{R,0}=\Omega_{k,0}=0$ (ΛCDM Model):
   In this case $\Omega_0 = \Omega_{NR,0}+\Omega_\Lambda = 1$

$$t = \frac{1}{H_0}\int_0^{\frac{a}{a_0}}\frac{xdx}{\sqrt{+\Omega_{NR,0}x\Omega_\Lambda x^4}} = \frac{1}{H_0}\frac{1}{\sqrt{\Omega_{NR,0}}}\int_0^{\frac{a}{a_0}}\frac{x^{\frac{1}{2}}dx}{\sqrt{1+\frac{\Omega_\Lambda}{\Omega_{NR,0}}x^3}} \tag{4.72}$$

With suitable substation we can solve this equation in the following form:

$$t = \frac{2}{3H_0}\frac{1}{\sqrt{\Omega_{NR,0}}}\sinh^{-1}\left[\left(\frac{\Omega_\Lambda}{\Omega_{NR,0}}\right)^{\frac{1}{2}}\left(\frac{a}{a_0}\right)\right] = \frac{2}{3H_0}\frac{1}{\sqrt{\Omega_{NR,0}}}\sinh^{-1}\left[\left(\frac{1-\Omega_{R,0}}{\Omega_{NR,0}}\right)^{\frac{1}{2}}\left(\frac{a}{a_0}\right)^{\frac{3}{2}}\right] \tag{4.73}$$

and age of the universe is

$$t_0 = \frac{2}{3H_0}\frac{1}{\sqrt{\Omega_{NR,0}}}\sinh^{-1}\left[\left(\frac{1-\Omega_{R,0}}{\Omega_{NR,0}}\right)^{\frac{1}{2}}\right] \tag{4.74}$$

Note that in this case parametric solutions are as follows:

$$a = a_0\left(\frac{\Omega_{NR,0}}{\Omega_\Lambda}\right)^{\frac{1}{3}}\sinh^{\frac{2}{3}}\theta \quad \text{and} \quad t = \frac{1}{3H_0}\frac{1}{\sqrt{\Omega_\Lambda}}\theta \tag{4.75}$$

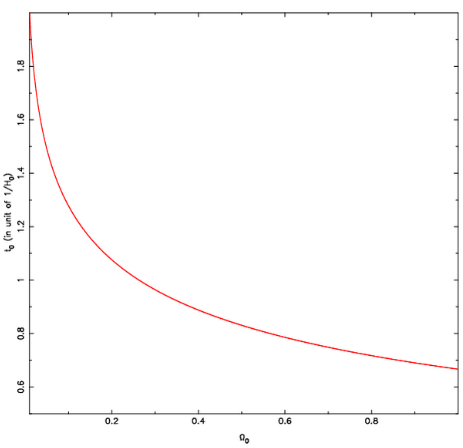


Figure 4.3: Age of the universe for ΛCDM case

Chapter 5

Clumpy Universe

5.1 Particle dynamics in an expanding universe

The homogeneous and isotropic expansion of the universe can be characterized by the cosmic scale factor $a(t)$. In general, it is more useful to work in a coordinate system called the comoving coordinate system which expands with the expansion of the universe.

Physical distance $r(t)$ and comoving distance $x(t)$ are related by the following way:

$$r(t) = a(t)x(t)$$

Note that in a smooth universe (perfectly homogeneous and isotropic) comoving distance $x(t)$ between two galaxies is constant. However, if there are inhomogeneities, as there are, then $x(t)$ is no longer a constant.

$$\dot{r}(t) = \dot{a}(t)x + a(t)\dot{x}$$

Here the first part $\dot{a}x = Hr$, is nothing but the homogeneous and isotropic expansion (hubble expansion) of the universe and the second part $a\dot{x}$ is called the *peculiar velocity* which is directly related to the inhomogeneities in the matter distribution in the universe.

In order to find the equation of motion for a particle in an expanding universe we can use Lagrangian mechanics. For a system of particles

$$L = \sum_{i=1}^{N} \left[\frac{1}{2}m_i\dot{r}_i^2 - m_i\Phi(r_i)\right] = \sum_{i=1}^{N} \left[\frac{1}{2}m_i(a\dot{x}_i + \dot{a}x_i)^2 - m_i\Phi_i\right]$$

$$L = \sum_{i=1}^{N} \left[\frac{1}{2}m_i a^2\dot{x}_i^2 - m_i\left(\Phi_i + \frac{1}{2}a\ddot{a}x^2\right)\right] + \frac{\partial\psi}{\partial t} = \sum_{i=1}^{N} \left[\frac{1}{2}m_i a^2\dot{x}_i^2 - m_i\phi_i\right] + \frac{\partial\psi}{\partial t}$$

where

$$\phi_i = \Phi_i + \frac{1}{2}a\ddot{a}x^2 \quad \text{and} \quad \psi = \frac{1}{2}a\dot{a}x^2$$

The last part of the above equation is total time derivative so we can neglect it.

From the above Lagrangian we can write the Euler Lagrangian equation of motion.

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

or

$$\ddot{x}_i + 2\frac{\dot{a}}{a}\dot{x}_i = -\frac{\nabla \phi_i}{a^2} \tag{5.1}$$

This equation can be written in terms of peculiar velocity $v_p = a\dot{x}$ also.

$$\dot{v}_i + \frac{\dot{a}}{a}v_i = -\frac{\nabla \phi_i}{a} \tag{5.2}$$

5.1.1 Canonical momentum

From the Lagrangian we can compute the canonical momentum p_i

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = ma^2\dot{x}_i = mav_i$$

Note that the source of the peculiar velocity is the following potential

$$\phi_i = \Phi_i + \frac{1}{2}a\ddot{a}x^2 = \Phi_i + \frac{1}{2}a^2\frac{\ddot{a}}{a}x^2$$

Now from the second Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G\rho}{3}$$

so

$$\phi_i = \Phi_i - \frac{2\pi G\rho}{3}a^2x^2$$

Note that the second part of the above potential is due to hubble expansion of the universe.

The source of potential ϕ_i are density inhomogeneities which are quantified in terms of a variable δ, called the *density contrast*, which is defined as follows:

$$\delta(x,t) = \frac{\rho(x,t) - \rho_b(t)}{\rho_b(t)} \tag{5.3}$$

In a homogeneous and isotropic universe density ρ at any point is a function of time only. In order to find ϕ_i we need to solve the Poisson equation.

$$\nabla_r^2\phi_{TOTAL} = 4\pi G\rho(x,t) = 4\pi G\rho_b(t) + 4\pi G\rho_b(t)\delta(x,t)$$

Since $\phi_{TOTAL} = \phi_b + \phi$ where $\nabla_r^2\phi_b = 4\pi G\rho_b(t)$, so the above equation can be written in the following form:

$$\nabla_x^2\phi = 4\pi Ga^2\rho_b\delta \tag{5.4}$$

In order to see how peculiar velocity v and canonical momentum p change with time let us consider the case of Einstein-de Sitter universe i.e., $a(t) \propto t^{\frac{2}{3}}$.

In this case if there are no inhomogeneities $\delta = 0$ and so $\phi_i = 0$.

$$\dot{v}_i + \frac{\dot{a}}{a} v_i = \dot{v}_i + \frac{2}{3t} v_i = 0$$

This gives the following solution:

$$v_i \propto t^{\frac{-2}{3}} \propto \frac{1}{a} \tag{5.5}$$

So in an Einstein-de Sitter case peculiar velocities decay as $\frac{1}{a}$, however, the canonical momentum $p = mav_i$ remains the same as the universe expands.

5.1.2 Discrete density field

In general we discretize density field in term of particles.

$$\rho(x,t) = \frac{1}{a^3} \sum_i m_i \delta_D^3(x - x_i(t)) \tag{5.6}$$

The background density

$$\rho_b(t) = \frac{1}{V} \int d^3x \rho(x,t) = \frac{1}{V} \int d^3x \frac{1}{a^3} \sum_i m_i \delta_D^3(x - x_i(t)) = \frac{M}{a^3 V}$$

Where

$$M = \sum_{i=1}^{N} m_i$$

Now we can solve the possion equation for this density.

$$\nabla^2 \phi = 4\pi G a^2 (\rho - \rho_b)$$

so

$$\phi(x) = -Ga^2 \int d^3x' \frac{\rho(x') - \rho_b}{|x' - x|}$$

and

$$\nabla \phi(x) = -Ga^2 \int d^3x' \frac{(\rho(x') - \rho_b)(\mathbf{x}' - \mathbf{x})}{|x' - x|^3}$$

Now if we put the value ρ w find

$$\nabla \phi_i = \frac{G}{a} \sum_{j \neq i}^{N} m_j \frac{x_j - x_i}{|x' - x|^3}$$

In terms of this, we can write equation of motion in the following form:

$$\dot{v}_i + \frac{\dot{a}}{a} v_i = -\frac{G}{a^2} \sum_{j \neq i}^{N} m_j \frac{x_j - x_i}{|x' - x|^3} \tag{5.7}$$

5.1.3 Tensor virial theorem

Let us consider a system of N gravitating particles.

Equation of motion for i'th particle

$$m_i \ddot{x}_i^\alpha = x_i^\alpha = m_i G \sum_{j\neq i} m_j \frac{x_j^\alpha - x_i^\alpha}{|x_j - x_i|^3}$$

Here subscripts are used for the labeling of particles and superscripts are tensor index. Multiplying both sides of the above equation by x_i^β and summing over all is.

$$\sum_i m_i x_i^\beta \ddot{x}_i^\alpha = G \sum_i m_i \sum_{j\neq i} m_j \frac{x_i^\beta (x_j^\alpha - x_i^\alpha)}{|x_j - x_i|^3}$$

Now

$$RHS = G \sum_i m_i \sum_{j\neq i} m_j \frac{x_i^\beta (x_j^\alpha - x_i^\alpha)}{|x_j - x_i|^3} = \frac{G}{2} \sum_i \sum_{j\neq i} m_j m_j \frac{(x_i^\alpha - x_j^\alpha)(x_i^\beta - x_j^\beta)}{|x_i - x_j|^3} = U^{\alpha\beta} \tag{5.8}$$

Here we have used

$$x_i^\beta = \frac{1}{2}(x_i^\beta - x_j^\beta) + \frac{1}{2}(x_i^\beta + x_j^\beta)$$

and the property that the multiplication of a symmetric and antisymmetric tensor is zero.

Now

$$LHS = \sum_i m_i x_i^\beta \ddot{x}_i^\alpha = \frac{d}{dt} \sum_i m_i x_i^\beta \dot{x}_i^\alpha - \sum_i m_i \dot{x}_i^\beta \dot{x}_i^\alpha$$

$$= \frac{d^2}{dt^2} \sum_i m_i x_i^\beta x_i^\alpha - \frac{d}{dt} \sum_i x_i^\alpha \frac{d}{dt}(m_i x_i^\beta) - \sum_i m_i \dot{x}_i^\beta \dot{x}_i^\alpha$$

$$= \frac{d^2}{dt^2} \sum_i m_i x_i^\beta x_i^\alpha - \frac{d}{dt} \sum_i m_i x_i^\alpha \dot{x}_i^\beta - \frac{d}{dt} \sum_i \dot{m}_i x_i^\alpha x_i^\beta - \sum_i m_i \dot{x}_i^\beta \dot{x}_i^\alpha$$

$$= \frac{d^2 I^{\alpha\beta}}{dt^2} - \frac{1}{2}\frac{d}{dt} \sum_i m_i (x_i^\alpha \dot{x}_i^\beta + x_i^\beta \dot{x}_i^\alpha) - M^{\alpha\beta} - 2T^{\alpha\beta}$$

$$= \frac{d^2 I^{\alpha\beta}}{dt^2} - \frac{1}{2}\frac{d}{dt} \sum_i m_i (x_i^\alpha \dot{x}_i^\beta + x_i^\beta \dot{x}_i^\alpha) - \frac{dM^{\alpha\beta}}{dt} - 2T^{\alpha\beta}$$

$$= \frac{d^2 I^{\alpha\beta}}{dt^2} - \frac{1}{2}\frac{d^2}{dt^2} \sum_i m_i x_i^\alpha x_i^\beta + \frac{1}{2}\frac{d}{dt} \sum_i \dot{m}_i x_i^\alpha x_i^\alpha - \frac{dM^{\alpha\beta}}{dt} - 2T^{\alpha\beta}$$

$$= \frac{1}{2}\frac{d^2 I^{\alpha\beta}}{dt^2} - \frac{1}{2}\frac{dM^{\alpha\beta}}{dt} - 2T^{\alpha\beta} \tag{5.9}$$

From the equation (5.8) and equation (5.9)

$$2T^{\alpha\beta} + U^{\alpha\beta} = \frac{1}{2}\frac{d^2 I^{\alpha\beta}}{dt^2} - \frac{1}{2}\frac{dM^{\alpha\beta}}{dt} \tag{5.10}$$

Where

$$T^{\alpha\beta} = \frac{1}{2}\sum_i m_i \dot{x}_i^\alpha \dot{x}_i^\beta, \qquad U^{\alpha\beta} = -\frac{G}{2}\sum_i \sum_{j\neq i} m_j m_j \frac{(x_i^\alpha - x_j^\alpha)(x_i^\beta - x_j^\beta)}{|x_i - x_j|^3}$$

$$I^{\alpha\beta} = \sum_i m_i x_i^\alpha x_i^\beta \quad \text{and} \quad M^{\alpha\beta} = \sum_i \dot{m}_i x_i^\alpha x_i^\beta$$

5.1.4 Virial theorem in an expanding universe

In an expanding universe equation of motion in tensor form is as follows:

$$m_i \ddot{x}_i^\alpha + 2\frac{\dot{a}}{a} m_i \dot{x}_i^\alpha = -\frac{1}{a}\frac{\partial \phi_i}{\partial x^\alpha}$$

where

$$\frac{\partial \phi_i}{\partial x^\alpha} = \frac{G}{a}\sum_j m_j \frac{(x_j^\alpha - x_i^\alpha)}{|x_j - x_i|^3}$$

Now multiplying the above equation of motion by x_i^β and taking sum over all is:

$$\sum_i m_i x_i^\beta \ddot{x}_i^\alpha + 2\frac{\dot{a}}{a}\sum_i m_i x_i^\beta \dot{x}_i^\alpha = -\frac{G}{a^3}\sum_j m_j \frac{x_i^\beta (x_j^\alpha - x_i^\alpha)}{|x_j - x_i|^3}$$

Apart from the second term in LHS and factor $\frac{1}{a^3}$, this equation is identical to equation of motion used in the last section. So after simplification:

$$2T^{\alpha\beta} + U^{\alpha\beta} = \frac{1}{2}\frac{d^2 I^{\alpha\beta}}{dt^2} - \frac{1}{2}\frac{dM^{\alpha\beta}}{dt} - 2\frac{\dot{a}}{a}\sum_i m_i x_i^\beta \dot{x}_i^\alpha \tag{5.11}$$

Now

$$2\frac{\dot{a}}{a}\sum_i m_i x_i^\beta \dot{x}_i^\alpha = \frac{\dot{a}}{a}\sum_i m_i (x_i^\beta \dot{x}_i^\alpha + x_i^\alpha \dot{x}_i^\beta) = \frac{\dot{a}}{a}\sum_i m_i \frac{d}{dt}(x_i^\alpha x_i^\beta)$$

$$= \frac{\dot{a}}{a}\frac{d}{dt}\sum_i m_i (x_i^\alpha x_i^\beta) - \sum_i \dot{m}_i (x_i^\alpha x_i^\beta) = \left(\frac{\dot{a}}{a}\right)\frac{dI^{\alpha\beta}}{dt} - M^{\alpha\beta}$$

So the equation (5.12) can be written as follows:

$$2T^{\alpha\beta} + U^{\alpha\beta} = \frac{1}{2}\frac{d^2 I^{\alpha\beta}}{dt^2} - \frac{1}{2}\frac{dM^{\alpha\beta}}{dt} - \left[\frac{\dot{a}}{a}\frac{dI^{\alpha\beta}}{dt} - M^{\alpha\beta}\right] \tag{5.12}$$

Where

$$U^{\alpha\beta} = -\frac{G}{2a^3}\sum_i \sum_{j\neq i} m_j m_j \frac{(x_i^\alpha - x_j^\alpha)(x_i^\beta - x_j^\beta)}{|x_i - x_j|^3}$$

$T^{\alpha\beta}, I^{\alpha\beta}$ and $M^{\alpha\beta}$ are the same as in the last section.

5.2 Some physical process

5.2.1 Jeans Process

Let us consider a system of molecules each of mass M in box of size L which are interacting with each other gravitationally as well as by thermal collisions.

Gravitational energy of the system

$$E_G = -\frac{GM^2}{L}$$

And thermal energy of the system

$$E_T = Nk_BT = \frac{M}{m}k_BT$$

Comparing these two energies

$$\frac{E_G}{E_T} = \frac{GMm}{Lk_BT} = \frac{GL^3\rho m}{Lk_BT} = \left(\frac{L}{L_J}\right)^2$$

Where

$$L_J = \sqrt{\frac{k_BT}{G\rho}} \tag{5.13}$$

From this expression it is clear that gravity dominates over thermal motions for $L > L_J$. This simply means that if we size of the system is smaller than a certain size (L_J) then the pressure of molecules is sufficient to prevent *gravitational collapse*. Let us compare the gravitational coll pase time scale t_G and press eure time scale t_P.

$$t_G = \sqrt{\frac{L}{g}} = \sqrt{\frac{L}{GM/L^2}} = \frac{1}{\sqrt{G\rho}}$$

$$t_p = \frac{L}{c_s} \quad \text{where} \quad c_s = \sqrt{\frac{P}{\rho}} = \sqrt{\frac{k_BT}{m}} \quad \text{is the speed of sound}$$

comaprinng these two time scales

$$\frac{t_G}{t_p} = \frac{1}{\sqrt{G\rho}}\frac{c_s}{L} = \frac{c_s/\sqrt{G\rho}}{L} = \frac{L_J}{L} \quad \text{where again} \quad L_J = \sqrt{\frac{k_BT}{G\rho}} \tag{5.14}$$

Now we can see that for $L < J_J, t_P < t_G$ which means gravitational collapse time scale is small in comparison to the time scale associated with pressure wave and vice versa.

5.2.2 Jeans mass in cosmology

It has been found that the in the present universe (z=0) density of non relativistic matter $\rho_0 = 10^{-28} kg \;\; mt^{-3}$ and temperature of cosmic background radiation $T_0 = 2.7K$. Since we know when we go back in time these quantity varies as $\rho \propto (1+z)3$ and $T \propto (1+z)$. Now we can compute the jeans length at *decoupling* ($z_{dec} \approx 1000$).

$$L_J(z = z_{dec}) = \sqrt{\frac{k_B T_{dec}}{G \rho_{dec} m}} = \left(\frac{1.4 \times 10^{-23} \times 3000.0}{6.67 \times 10^{-11} \times 1.4 \times 10^{-19} \times 10^{-27}} \right)^{\frac{1}{2}} = 1.6 \times 10^{18} mt = 50pc$$

Now at decoupling $\rho = 1.4 \times 10^{-19} = 2M_{\odot} pc^{-3}$ so the jeans mass at decoupling $M_j = \rho L_j^3 = 2M_{\odot} pc^{-3} \times (50pc)^3 = 3 \times 10^5 M_{\odot}$, and this is the typical mass of a globular cluster in the present universe.

5.2.3 Cosmological perturbations

In general there are following two type of density perturbations in the universe. If we assume that there is only radiation and matter in the universe then if we perturb the energy density of only matter then the perturbations are said to be *isothermal* and if we perturb the both then perturbations are called *adiabatic*.

	Matter perturbations	Radiation perturbations
Isothermal	Yes	No
Adiabatic	Yes	Yes

5.2.4 Isothermal perturbations

Let us consider a mixture of electrons and photons and compare the strength of gravitational and interaction between electrons and drag acting on electrons due to Thompson scattering.

Force per unit mass due to gravity

$$F_G = \frac{4\pi}{3} G \rho R$$

and drag acting due to Thomson scattering

$$F_{drag} = \frac{aT^4 (\sigma_T v)}{m_H c}$$

Here σ_T is the scattering cross section for Thomson scattering, v is the velocity dispersion of electrons, m_H is the mass of the protons, c is the speed of light and a is the stephan-Boltzmann constant.

Now comparing these two:

$$\frac{F_{drag}}{F_G} = \frac{3a\sigma_T v T^4}{4\pi G R m_H c \rho} = \left(\frac{3a\sigma_T v T_0^4 (1+z)^4}{4\pi G R m_H c \rho_0 (1+z)^3} \right) = \left(\frac{3a\sigma_T}{4\pi G m_H c} \right) \left(\frac{T_0^4}{\rho_0} \right) \left(\frac{v(1+z)}{R} \right)$$

Now for an *Einstein-de Sitter model*

$$R = \frac{2}{3H_0}\frac{v}{(1+z)^{3/2}}$$

So

$$\frac{F_{drag}}{F_G} = \frac{9\sigma_T a}{8\pi G m_H c}\left(\frac{T_0^4 H_0}{\Omega_0 \rho_c}\right)(1+z)^{5/2} \approx 10^{-8}(1+z)^{5/2}$$

From this expression it is clear that for $z > 1500$ gravitational collapse is halted by the pressure provided by photons.

5.2.5 Adiabatic perturbations: random walk of photons

Again consider a mixture of electrons and photons for which we can define the *mean free path* (λ) as

$$\lambda = \frac{1}{n_e \sigma_T}$$

here n_e is the number density of electrons.

Now let us compare the time scale t_{dif} needed for photons to travel a distance R with the age of the universe t_{age}.

From the calculation of random walk we know distance traveled D_n in N steps is proportional to $\sqrt{N}$ so time diffusion time

$$t_{dif} = \text{ no of steps required to travel the distance } R \times \text{time taken in one step}$$

$$t_{dif} = \left(\frac{R}{\lambda}\right)^2\left(\frac{\lambda}{c}\right) = \frac{R^2\lambda}{c} = \frac{R^2}{c n_e \sigma_T}$$

Now for an Einstein-de Sitter universe

$$t_{age} = \frac{2}{3H_0}(1+z)^{-3/2}$$

So

$$\frac{t_{diff}}{t_{age}} = \frac{R^2}{\lambda c}\frac{3H_0}{2}(1+z)^{3/2} = \frac{R^2}{R_c^2} \tag{5.15}$$

From this expression it is clear that for

$$R^2 > R_s^2 = \frac{2c}{3H_0}\lambda(1+z)^{-3/2}$$

diffusion time scale is large in comparison to the age of the universe so diffusion is ineffective to was out the fluctuation of size $R > R_s$. Let us compute the mass associated with R_s.

$$M_s = \frac{4\pi R_s^3}{3}\rho = \frac{4\pi R_s^3}{3}\left(\frac{1}{\rho_c \Omega_0}\right)^{1/2}(1+z)^{-15/2}$$

5.3 Expansion of the universe

Expansion of a homogeneous and isotropic universe is governed by the Friedman equations:

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} \tag{5.16}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} + \frac{\Lambda}{3} \tag{5.17}$$

We can write these equations in term of the following variables:

$$H = \frac{\dot{a}}{a}, \quad \Omega = \frac{8\pi G\rho}{3H^2} \text{ and } \Omega_\Lambda = \frac{\Lambda}{8\pi G}, \quad \Omega_{TOTAL} = \Omega + \Omega_\Lambda$$

Writing the first equation today $t = t_0$ gives us the value of k in terms of present Ω_{TOTAL}

$$k = H_0^2 a_0^2 (\Omega_{TOTAL}^0 - 1)$$

Putting this value in the first equation we find:

$$H^2 = \frac{\dot{a}^2}{a^2} = H_0^2 \left[\Omega_{TOTAL} - (\Omega_{TOTAL}^0 - 1) \left(\frac{a_0}{a}\right)^2 \right] \tag{5.18}$$

Now we can split Ω_{TOTAL} in three parts

$$\Omega_{TOTAL} = \Omega_{NR} + \Omega_R + \Omega_\Lambda$$

Here NR and R subscripts are used for non relativistic and relativistic matter.

Now we can use the following fact

$$\Omega_{NR}(a) = \Omega_{NR}(a_0) \left(\frac{a_0}{a}\right)^3 \quad \Omega_R(a) = \Omega_{NR}(a_0) \left(\frac{a_0}{a}\right)^4 \quad \Omega_\Lambda(a) = \Omega_\Lambda(a_0) \tag{5.19}$$

Now in what follows we will use the following convention

$$\Omega_{NR}(a_0) = \Omega_{NR}, \quad \Omega_R(a_0) = \Omega_R, \quad \Omega_\Lambda(a_0) = \Omega_\Lambda, \quad \text{and } \Omega_{TOTAL} = \Omega$$

From equation (5.18) and equation (5.19)

$$H = H_0 \left[\Omega_{NR} \left(\frac{a_0}{a}\right)^3 + \Omega_R \left(\frac{a_0}{a}\right)^4 + (1 - \Omega) \left(\frac{a_0}{a}\right)^2 + \Omega_\Lambda \right]^{\frac{1}{2}} = H_0 f(a) \tag{5.20}$$

where

$$f(a) = \left[\Omega_{NR} \left(\frac{a_0}{a}\right)^3 + \Omega_R \left(\frac{a_0}{a}\right)^4 + (1 - \Omega) \left(\frac{a_0}{a}\right)^2 + \Omega_\Lambda \right]^{\frac{1}{2}}$$

5.3.1 Hubble length

In structure formation theories there is a great importance of the *hubble distance* (d_H), which is defined by the distance beyond which galaxies recede with superluminal speeds. No physical process can operate over length scales larger than d_H

$$d_h = \frac{c}{H} = \frac{c}{H_0}\frac{1}{f(a)} \tag{5.21}$$

Let us see how d_H vary with the expansion for the following simple cases:

- Matter dominated flat universe

  For this case $\Omega_{NR} = \Omega = 1$ so

  $$f(a) = \left(\frac{a_0}{a}\right)^{\frac{3}{2}}$$

  and so

  $$d_h = \frac{c}{H_0}\left(\frac{a}{a_0}\right)^{\frac{3}{2}}$$

- Radiation dominated flat universe

  For this case $\Omega_R = \Omega = 1$ so

  $$f(a) = \left(\frac{a_0}{a}\right)^{2}$$

  and so

  $$d_h = \frac{c}{H_0}\left(\frac{a}{a_0}\right)^{2}$$

- Λ dominated flat universe

  For this case $\Omega_\Lambda = \Omega = 1$ so

  $$f(a) = 1$$

  and so

  $$d_h = \frac{c}{H_0}$$

5.3.2 Particle horizon

Sometime people confuse the hubble length d_H with the particle horizon r_H which is defined as follows:

$$r_h = a\int_0^t \frac{cdt}{a} = a\int_0^t \frac{cda}{Ha^2} = \frac{c}{H_0}a\int_0^t \frac{da}{a^2 f(a)} \tag{5.22}$$

Let us compute the value of r_H for the same three simple cases:

- Matter dominated flat universe For this case $\Omega_{NR} = \Omega = 1$ so

  $$f(a) = \left(\frac{a_0}{a}\right)^{\frac{3}{2}}$$

so

$$r_H = \frac{c}{H_0} a \int_0^t \frac{a^{\frac{-1}{2}} da}{a_0^{\frac{3}{2}}} = \frac{2c}{H_0} \left(\frac{a}{a_0}\right)^{\frac{3}{2}} = 2d_H$$

- Radiation dominated flat universe

  For this case $\Omega_{NR} = \Omega = 1$ so

$$f(a) = \left(\frac{a_0}{a}\right)^2$$

so

$$r_H = \frac{c}{H_0} a \int_0^t \frac{da}{a_0^2} = \frac{c}{H_0} \left(\frac{a}{a_0}\right)^2 = d_H$$

- Λ dominated flat universe

  For this case $\Omega_{Lambda} = \Omega = 1$ so

  Since in this case $f(a) =$ constant, so

$$r_H = \frac{c}{H_0} a \int_0^t \frac{da}{a^2} = \frac{c}{H_0} = d_H$$

5.3.3 Physical length scale

In an expanding universe any physical length λ change with the expansion by the following way:

$$\lambda \propto a \tag{5.23}$$

In the present universe energy density of non relativistic matter is very large in comparison to the energy density of relativistic matter i.e., $\Omega_{NR} \approx 150,000\Omega_R$. However, if we go back in time then energy density of radiation grows with faster rate in comparison to matter, so there comes a times, called the *time of matter-radiation equality* (t_{eq}), before which the universe was radiation dominated.

At matter-radiation equality

$$\Omega_R(a_{eq}) = \Omega_{NR}(a_{eq})$$

or

$$\Omega_{NR} \left(\frac{a_0}{a_{eq}}\right)^3 = \Omega_R \left(\frac{a_0}{a_{eq}}\right)^4$$

or

$$\frac{a_{eq}}{a_0} = \frac{\Omega_R}{\Omega_{NR}} \approx 6.67 \times 10^{-5}$$

This means that when the size of the universe was 10^{-5} times smaller than its present size then it was radiation dominated.

As has been discussed that the physical length scales vary $\frac{1}{a}$ when the universe expands, however, hubble length d_H in matter and radiation dominated universe vary as $a^{\frac{3}{2}}$ and a^2 respectvely. It tern out that all length scales of cosmological importance were larger than the hubble length in very early universe. These scales are said to be entering hubble region when they become equal to hubble length. Note that small scales enter hubble region before the large scales (see Figure (5.1)). Fate of density perturbation of some particular length scale depends on when it enters the hubble region e.g., if it enters in radiation dominated era then there is less growth in comparison to when it enters in matter dominated era. Note that in what follows we will use a wrong but popular nomenclature *horizon entry* for the *hubble entry*.

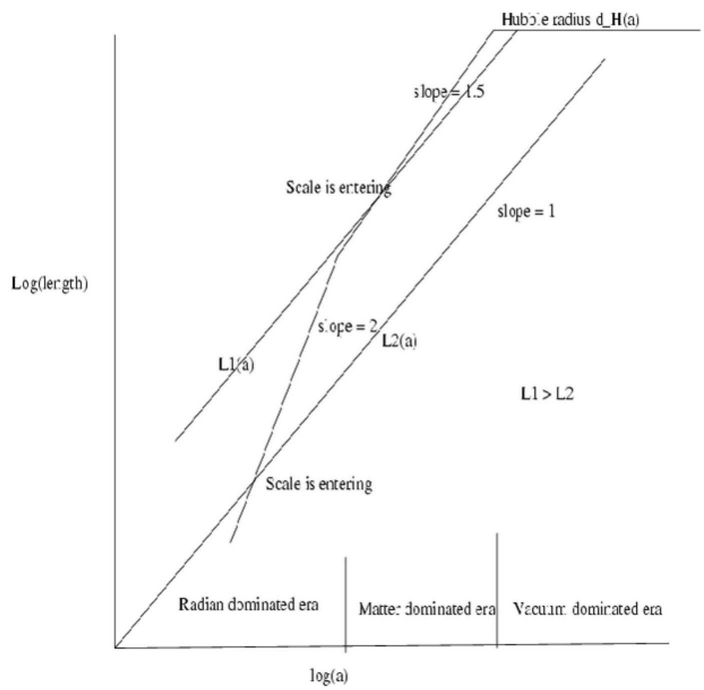


Figure 5.1: Various length scales enter the horizon in different regime. Large length scales enter later and small earlier

We can find a useful relation for the horizon entry for a mode of wavelength λ. At very early epoch we can ignore the curvature term in Friedmann equations so the size of horizon

$$d_H^2 = \frac{a^2}{\dot{a}^2} = \frac{3}{8\pi G\rho}$$

This gives

$$\frac{dln d_H}{dt} = -\frac{1}{2}\frac{dln\rho}{dt}$$

Since we know

$$d(\rho a^3) = -P da^3$$

or

$$dln\rho = -3\left(1 + \frac{P}{\rho}\right)$$

Using this we find

$$\frac{dlnd_H}{dt} = \frac{3}{2}\left(1+\frac{P}{\rho}\right)\frac{dln\rho}{dt} \tag{5.24}$$

Using equation (5.23) and equation (5.24)

$$\frac{dlnd_H}{dln\lambda} = \frac{3}{2}\left(1+\frac{P}{\rho}\right) \tag{5.25}$$

This means that as long as $P > 0$, RHS is greater than unity and so d_H **grows as** $\lambda^n; n > 1$. However, if $P < 0$ (like in the case of Λ), d_H can decrease with λ.

5.4 Probability

Corresponding to every physical process we can consider a set Ω of all possible outcomes. A typical element of this set can be represented by ω. The subset $\mathcal{B}$ of Ω i.e., $\mathcal{B} \subset \Omega$ which contains all such outcomes with whom we can assign probability, which is defined as below, is called the *Borel Space*.

Probability $P(A)$ is a real number which we can assign with an outcome A. It has to satisfies the following conditions:

- $$P(A) \geq 0 \;\; \text{for all} \;\; A \in \mathcal{B}$$

- $$P(\Omega) = 1$$

- $$P(\cup A_i) = \sum_i P(A_i)$$

Some of other interesting properties of probability are as follows:

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

Note that $P(A_1 \cap A_2)$ can be written as $P(A_1, A_2)$ also.

5.4.1 Conditional Probability

If A and B are two events then the conditional probability of B for a given A is defined as follows:

$$P(A|B) = \frac{P(A,B)}{P(B)}$$

Example 1: Let consider a fair die and $A = \{2\}, B = \{2,4,6\}$.

For this case

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/6}{1/2} = \frac{1}{3}$$

Bayes' Theorem: Since

$$P(A,B) = P(A|B)P(B) = P(B|A)P(A)$$

So we can find

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)} \tag{5.26}$$

Bayes' theorem plays an important role in parameter estimation. It says the *posterior probability* $P(p|x,X)$ of a parameter p for having certain values, from a observational data x under the hypothesis X, is proportional to the probability $P(x|p,X)$ (this probability is called the *Likelihood function*)of having the data x, with given input parameter p under the same hypothesis X.

$$P(p|x,X) = \frac{P(p|x,X)P(p|X)}{P(x|X)} \tag{5.27}$$

Here $P(p|X)$ is the a *prior probability* of the parameter p, given the prior assumption X and the probability $P(x|X)$ is used just for normalization which ensure that the integral over the *posterior probability* is unity.

5.4.2 Maximum Liklihood

Liklihood function $\mathcal{L}$ is defined as follows:

$$\mathcal{L} = \Pi_{i=1,N} p_i \tag{5.28}$$

Where *pi* is the probability density. For example it could be Gaussian. In that case

$$p(x_i) = \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)$$

An *estimator* of a parameter p is defined by $\hat{p}$ is said to be unbiased if the expectation value of the estimates equals to true value.

$$<\hat{p}> = \int \hat{p}\mathcal{L}(x|P)dx$$

There is a theorems which says an unbiased estimator is the *maximum liklihood estimator* if follows certain condition (Cramer-Rao bound exists).

The value of $\hat{p}$ for which liklihood attains its maximum values, given the observed data, can be obtained by:

$$\frac{\partial \mathcal{L}}{\partial p_\alpha}|_{p=\hat{p}_\alpha} = 0$$

Note that such an estimator which obeys Cramer-Rao bound exists if the liklihood function $\mathcal{L}$ is gaussian about its maximum. The central limit theorem says that $\mathcal{L}$ is asymptotically Gaussian in the limit of a large amount of data.

Maximum liklihood method is frequently used for parameter estimation. In order to see how it works let us consider the following example.

We toss a coin 80 times and get head 49 times. Now if the coin was among three coins, out of which two are unfair and we do not know which of the coin we have. Now if the probability of getting head for the coins are $\frac{1}{3}, \frac{1}{2}$ and $\frac{2}{3}$ then using the method of maximum liklihood find that which of the coin we have.

Now the liklihood functions of the three coins are as follows:

$$P(\text{ for } \frac{1}{3}) = \frac{80!}{49!\ 31!}\left(\frac{1}{3}\right)^{49}\left(1-\frac{1}{3}\right)^{31} = 0.00$$

$$P(\text{ for } \frac{1}{2}) = \frac{80!}{49!\ 31!}\left(\frac{1}{2}\right)^{49}\left(1-\frac{1}{2}\right)^{31} = 0.012$$

$$P(\text{ for } \frac{2}{3}) = \frac{80!}{49!\ 31!}\left(\frac{2}{3}\right)^{49}\left(1-\frac{2}{3}\right)^{31} = 0.054$$

From this it is clear that the liklihood (P) is maximum for the third case so this is the value which maximize the liklihood or the most probable case.

5.4.3 Probability distribution

A function $P(\lambda)$ which define the probabilities $P(\{x \leq \lambda\})$ for the sets $\{x \leq \lambda\}$ for all λ is called the *probability distribution function*. This function satisfies the following:

$$\lambda \longrightarrow +\infty : P(\lambda) \longrightarrow P(\Omega) = 1$$

$$\lambda \longrightarrow -\infty : P(\lambda) \longrightarrow P(\phi) = 0$$

In case $P(\lambda)$ is differentiable, we can define a function $\rho(\lambda)$ called the *probability density*.

$$\rho(\lambda) = \frac{dP(\lambda)}{d\lambda} \quad \text{or} \quad P(\lambda) = \int_{-\infty}^{\lambda} \rho(x)dx$$

With the following normalization.

$$\int_{-\infty}^{\infty} \rho(x)dx = 1$$

5.4.4 Random Variables

A discrete random variable is a collection of possible elementary events together with their probabilities.- Honerkamp (Statistical physics, Springer)

Note that in place discrete set, the possible realizations or outcomes can form a continiuos set in $\mathcal{R}$. In this case it is not possible to have a set of probabilities for all outcomes. In this case, in place of probability it is more useful to define probability distribution $P_X(\lambda)$ or probability density $\rho_X(x)$. Among various type of random distribution, the following two are more important.

1. Uniform Random: In this case the probability density $\rho_X(x)$ is a constant.

$$\rho_X(x) = \begin{cases} 1, & \text{if} \quad 0 \le x \le 1 \\ 0, & \text{Otherwise} \end{cases} \tag{5.29}$$

As an example let us consider a set of all possible realizations of X being a real numbers in the interval (a, b) with uniform probability then

$$\rho_X(x) = \frac{1}{b-a} \tag{5.30}$$

Probability of finding $x_1 \le x \le x_2$ can be given by the following:

$$P(\{x_1 \le x \le x_2\}) = \int_{x_1}^{x_2} \rho_X(x)dx$$

2. Gaussian Random: In this case the probability density is given by the following expression:

$$\rho_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \tag{5.31}$$

Here μ and σ are the mean and variance of the distribution respectvely.

We can generate Gaussian random numbers from uniform random numbers with the help of following expression:

$$y_1 = \sqrt{(-2ln(x_1))}\cos(2\pi x_2) y_2 = \sqrt{(-2ln(x_1))}\sin(2\pi x_2) \tag{5.32}$$

Here x_1, x_2 are two uniform random numbers and y_1, y_2 are two Gaussian random numbers.

5.4.5 Moments

Once we have a given a probability distribution, we can find various statistical measures. Some of them are as following.

1. Expectation Value:- Expectation values of function $f(x)$ of some random variable x which have probabity density $\rho(x)$ is defined as following:

$$E(f(x)) =< f(x) >= \int d^n x f(x)\rho(x) \tag{5.33}$$

2. Mean:- Mean μ of a random number X is defined as

$$< X >= \int dx x\rho(x) = \mu (\text{for normal or Gaussian distribution}) \tag{5.34}$$

3. Moments:- The m'th moment of a random number x is defined as

$$< X^m >= \int dx x^m \rho(x) \tag{5.35}$$

Note that for $m = 1$, it gives the mean.

In general, moments are defined around the mean μ.

$$< (X-\mu)^k >= \begin{cases} 0, & \text{if } k \text{ is odd} \\ 1.3.....(k-1).\sigma^k, & \text{if } k \text{ is even} \end{cases} \tag{5.36}$$

4. Variance:- This is defined as follows:

$$Var(X) =< (X- < X >)^2 >= \int dx(x- < X >)^2 \rho(x) \tag{5.37}$$

For normal or Gaussian distribution $Var(X) = \sigma^2$.

5. Covariance In the case of a multivariate distribution we can define second moments with respect to different components

$$< X_i X_j >= \int d^n x x_i \; x_j \; \rho(x_1, x_2, ..., x_n) \tag{5.38}$$

and the covariance matrix is defined as

$$Cov(X_i, X_j) = \sigma^2_{ij} =< (X-\mu)_i (X-\mu)_j > \tag{5.39}$$

6. Cumulants:- We can define an important functions $G(k)$ called the *characteristic function* by the following way:

$$G(k) =< e^{ikX} >= \int dx e^{ikX} \rho(x) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} < X^n > \tag{5.40}$$

With the help of this function, we can define *cumulants* by the following way.

$$ln \; G(k) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} k_n$$

Some of the cumulants are as follows:

$$k_1 = \mu =< X >$$

$$k_2 = Var(X) =< X^2 > - < X >^2$$

$$k_3 =< X^3 > -3 < X^2 >< X > +2 < X >^3 \tag{5.41}$$

Note that for a Gaussian distribution

$$G(k) = e^{\left(i\mu k - \frac{1}{2}\sigma^2 k^2\right)}$$

and we find

$$k_1 = \mu, k_2 = \sigma^2 \;\; \text{other higher cumulants are zero}$$

5.4.6 Center Limit Theorem

Before going to discuss the *center limit* theorem, it is useful to mention the following properties of random variables.

- If X_1, X_2 and Z are three independent variables such that

$$Z = X_1 + X_2$$

then the probability distribution Z is the convolution of the individual densities of X_1 and X_2 i.e.,

$$\rho_Z(z) = \int dx_1 \rho_{X_1}(x_1)\rho_{X_2}(z - x_1) \tag{5.42}$$

Note that in this case

$$G_Z(k) = G_{X_1}(k)G_{X_2}(k)$$

$$< Z >=< X_1 > + < X_2 >$$

$$Var(Z) = Var(X_1) + Var(X_2)$$

- If X_1, X_2 and Z are three independent variables such that

$$Z = X_1 X_2$$

then

$$\rho_Z(z) = \int dx_1 dx_2 \rho_{X_1}(x_1)\frac{1}{|x_1|}\rho_{X_2}\left(\frac{z}{x_1}\right)$$

Center Limit Theorem (CLT)

If $X_1, X_2, ..., X_N$ are independent and identically distributed random numbers with

$$< X_i >= 0$$

and

$$Var(X_i) = \sigma^2, \quad for \quad i = 1, ..., N$$

then if we define

$$Z_N = \frac{1}{\sqrt{N}}(X_1 + X_2 + ... + X_N)$$

the it follows that

$$< Z_N >= 0$$

and

$$Var(Z_N) = \frac{1}{-}\sum_{i=1}^{N} Var(X_i) = \sigma^2$$

and all higher moments and cumulants decrease as fast as $\frac{1}{\sqrt{N}}$ for $N \to \infty$.

The most important consequence of the CLT is that *as* $N \to \infty$ *the random number* Z_N *is a Gaussian random number with mean* 0 *and variance* σ^2.

5.4.7 Convolution

Convolution $h(x)$ of two functions $f(x)$ and $g(x)$ of two functions is defined as follows:

$$h(x) = f \star g = \int f(x')g(x-x')dx' \tag{5.43}$$

There is one important property of convolution.

$$h_k = f_k \; g_k \tag{5.44}$$

Here f_k, g_k and h_k are the fourier transforms of the function $f(x), g(x)$ and $h(x)$ respectvely. This can be proves as follows:

$$\begin{aligned} h(x) = f(x) \star g(x) &= \int d^3y f(y) g(x-y) \\ &= \int d^3y f(y) \int d^3k g_k e^{-ik.(x-y)} \\ &= \int d^3k g_k \left(\int d^3y f(y) e^{-ik.y} \right) e^{-ik.x} \\ &= \int d^3k g_k f_k e^{-ik.x} \end{aligned}$$

but we know that

$$h(x) == \int d^3k h_k e^{-ik.x}$$

This means that

$$h_k = g_k \; f_k \tag{5.45}$$

5.5 Cosmological density fields

5.5.1 Power spectrum

As has been mentioned that cosmological density perturbations are given in term of the density contrast.

$$\delta(x,t) = \frac{\rho(x,t) - \rho_b(t)}{\rho_b(t)} \tag{5.46}$$

We can write it in terms of Fourier transforms

$$\delta(x,t) = \frac{V}{(2\pi)^3} \int d^3k \delta_k(t) e^{-i \; \mathbf{k.x}} \tag{5.47}$$

Where δ_k is the density contrast in Fourier space:

$$\delta_k(t) = \frac{1}{V} \int d^3x \delta(x,t) e^{i\mathbf{k.x}} \tag{5.48}$$

Note that here δ as well as δ_k are dimensional physical quantities. In what follows we will be mainly discussing δ and δ_k at the same time so we can write $\delta(x,t) = \delta(x)$ and $\delta_k(t) = \delta_k$ for convinence.

From *Parseval theorem*:

$$\frac{V}{(2\pi)^3}\int d^3k|\delta_k|^2 = \frac{1}{V}\int d^3x|\delta(x)|^2 \tag{5.49}$$

Here the quantity $|\delta_k|^2 = P(k)$, is called the *Power Spectrum* .

We can find a relation between correlation between two frequencies i.e., $< \delta_k\delta_k' >$ and correlation between density contrasts between two length scales $< \delta(x)\delta(y) >$, by the following way:

$$< \delta_k\delta_{k'} >= \left\langle \frac{1}{V}\int d^3y\delta(y)e^{i\mathbf{k}.\mathbf{y}}\frac{1}{V}\int d^3x\delta(x)e^{i\mathbf{k}'.\mathbf{x}} \right\rangle$$

$$\frac{1}{V}\int d^3y\frac{1}{V}\int d^3x < \delta(x)\delta(y) > e^{i\mathbf{k}.\mathbf{y}}e^{i\mathbf{k}'.\mathbf{x}}$$

Let us substitute $y = x + r$ (Homogeneity of the universe)

$$< \delta_k\delta_{k'} >= \frac{1}{V}\int d^3re^{i\mathbf{k}.\mathbf{r}} < \delta(x+r)\delta(x) > \frac{1}{V}\int d^3xe^{i(\mathbf{k}+\mathbf{k}').\mathbf{x}}$$

$$< \delta_k\delta_{k'} >= (2\pi)^3\delta_D(\ \mathbf{k}+\mathbf{k}')\frac{1}{V}\int d^3re^{i\mathbf{k}.\mathbf{r}}\xi(r) = (2\pi)^3\delta_D(\ \mathbf{k}+\mathbf{k}')P(k) \tag{5.50}$$

Here

$\xi(r) =< \delta(x+r)\delta(x) >$ is called the *two point correlation function*

$\delta_D(\ \mathbf{k}+\mathbf{k}') = \frac{1}{(2\pi)^3}\int \frac{d^3x}{V}e^{i(\mathbf{k}+\mathbf{k}').\mathbf{x}}$ is the usual Dirac delta function.

$P(k) = \frac{1}{V}\int d^3re^{i\mathbf{k}.\mathbf{r}}\xi(r)$ is power spectrum

Now we can generalize the relation (5.51) in the following form:

$$< \delta_{k_1}\delta_{k_2}...\delta_{k_n} >= (2\pi)^{3n/2}\delta_D(k_1+k_2...+k_n)P_N(k_1,k_2...k_n) \tag{5.51}$$

Here $P_2(k_1,k_2)$ is the usual power spectrum and $P_3(k_1,k_2,k_3)$ is called the *bispectrum*. Note that the main reason behind the above relation is *homogeneity* of the space.

Note that our $\delta(x)$ and δ_k are dimensionless quantities so is the two point correlation function $\xi(r)$, however, power spectrum $P(k)$ is not a dimensionless quantity because:

$$< |\delta_k|^2 >= (2\pi)^3\delta_D^3(k+k')P(k) = \text{ dimensionless}$$

So the dimensions of $P(k)$ are dimension of $1/\delta_D^3(k+k')$ which are the dimensions of volume.

We can define a dimensionless power spectrum $\Delta^2(k)$ by the following way:

$$\Delta^2(k) = \frac{k^3}{(2\pi)^2} P(k) \tag{5.52}$$

Most of the times power spectrum generally means the power spectrum of density contrast δ, however, some time we need to use the power spectrum of potential (P_ϕ) also, which can de defined as follows:

Possion equation in comoving coordinates can be written as

$$\nabla^2 \phi = 4\pi G \rho_b a^2 \delta$$

or in Fourier space

$$-k^2 \phi_k = 4\pi G \rho_b a^2 \delta_k$$

$$P_k(\phi) =< |\phi_k|^2 >= 4\pi G \rho_b a^2 \frac{< |\delta_k|^2 >}{k^4} = 4\pi G \rho_b a^2 \frac{P_k(\delta)}{k^4} \tag{5.53}$$

If we consider a power law model i.e., $P_k(\delta) \propto k^n$ then

$$P_k(\phi) \propto k^{n-4}$$

and

$$\Delta^2_\phi(k) \propto k^{n-1}$$

If $P(k) \propto k$ then above equation gives

$$\Delta^2_\phi = constant$$

This particular choice of $P(k)$ is called *Harrison -Zeldovich* power spectrum which is also called *scale invariant power spectrum* because it gives equal power at all scales.

5.5.2 Correlation functions

In the hierarchy of N-point correlation function, the simplest is the two point correlation function ξ_2 is the two point correlation, which can be defined by the following ways:

- Conditional probability of finding a particle in volume δV_2 around point r_2, given that there is a particle in volume δV_1 around the position r_1

$$P(2|1) = \bar{n} \delta V_2 (1 + \xi_2(r_{12})) \tag{5.54}$$

- Probability of finding a particle in volume δV_1 around position r_1 and another in volume δV_2 around r_2

$$dP_{12} = \bar{n}^2 \delta V_1 \delta V_2 (1 + \xi_2(r_{12})) \tag{5.55}$$

- Excessive probability over random noise

  If we consider a random distribution of particle with average density $\bar{n}$ then the probability of having a particle in voule δV will be

  $$dP = \bar{n}\delta V$$

  However, if there are correlations, then the probability of having another particle in the same volume δV at r distance from the second will be

  $$dP = \bar{n}\delta V(1 + \xi_2(r))$$

The higher order correlation function can be defined in terms of the **connected part of the joint ensemble average of the density in arbitrarily number of points**.

$$\xi_N(x_1, x_2, ..., x_N) =< \delta(x_1)\delta(x_2)...\delta(x_N) >_c \tag{5.56}$$

Here connected part is define as follows:

$$< \delta(x_1)\delta(x_2)...\delta(x_N) >$$

$$=< \delta(x_1)\delta(x_2)...\delta(x_N) >_c + \sum \mathcal{S} \in \mathcal{P}(\{x_1, x_2, ..., x_n\})\Pi_{s_i \in \mathcal{S}}\xi_{ns_i}(x_{s_i(1)}, ...x_{s_i(ns_i)}) \tag{5.57}$$

This decomposition into connected and disconnected (irreducible and irreducible) parts form some of the cases this is as following:

$$< (1, 2) >=< (1) >< (2) > + < (1, 2) >_c$$

$$< (1, 2, 3) >=< (1) >< (2) >< (3) > + < (1, 2) >< (3) >$$
$$+ < (1) >< (2, 3) > + < (1, 3) >< (2) > + < (1, 2, 3) >_c$$

Here $< (1, 2) >=< \delta(x_1)\delta(x_2) >$ etc.
We define moments of the connected part in the following ways:

$$< \delta >_c =< \delta >$$

$$< \delta^2 >_c =< \delta >^2 - < \delta >_c^2 = \sigma^2$$
$$< \delta^3 >_c =< \delta >^3 - 3 < \delta >_c^2 < \delta_c - < \delta >_c^3$$

Three point correlation function $\xi_4(x_1, x_2, x_3)$ can be defined in terms of the probability dP_{123} of finding a particles 1, 2, and 3, in volumes in $\delta V_1, \delta V_2$ and δV_3 around the position x_1, x_2 and x_3 respectvely.

$$dP_{123} = \bar{n}^3\delta V_1\delta V_2\delta V_3(1 + \xi_2(1, 2) + \xi_2(2, 3) + \xi_2(3, 1) + \xi_3(1, 2, 1))$$

Here $\xi_3(1, 2, 1)) =< \delta(x_1)\delta(x_2)\delta(x_3) >_c$, is the *three point correlation function*. If any one of the pair in three particles is separated by infinite distance then ξ_3 becomes zero.

5.5.3 Variance

In general we console density contrast $\delta(x)$ with a window function and call the convolved function *filtered density contrast* $\delta_R(x)$.

$$\delta_W(x) = \int \delta(x')W(x-x')d^3x' \tag{5.58}$$

There are two popular choice of for the window function: (1) Spherical top hat and (2) Gaussian.

Spherical top hat window function is defined by the following function:

$$W(R, x-x') = \begin{cases} 1, & \text{if} \quad |x-x'| < R \\ 0, & \quad \text{Otherwise} \end{cases} \tag{5.59}$$

We can compute the Fourier transform of this function also.

$$W_k(R) = \frac{1}{V}\int_0^R d^3x e^{i\ \mathbf{k.x}} = \frac{1}{V}\int_0^R x^2 dx \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta e^{ikx\cos\theta}$$

$$W_k(R) = \frac{3}{4\pi R^3}\frac{4\pi}{k^3}(\sin KR - kR\cos kR) = 3\left(\frac{\sin kR - KR\cos kR}{(kR)^3}\right) \tag{5.60}$$

$$\delta_R(x) = \int \delta(x')W(R, x'-x)d^3x'$$

And so from the property of convolution functions

$$\delta_k(R) = \delta_k W_k(R)$$

Note that

$$< \delta_R(x) >= 0$$

and

$$< (\delta_R(x))^2 >= \sigma^2(R) = \frac{1}{(2\pi)^3}\int d^3k|\delta_k(R)|^2 = \frac{1}{(2\pi)^3}\int d^3k|\delta_k|^2|W_k(R)|^2$$

$$= \int \frac{dk}{k}\left(\frac{k^3P(k)}{2\pi^2}\right)|W_k(R)|^2 = \int dlnk \ \ \Delta^2(k)|W_k(R)|^2 \tag{5.61}$$

Now using the spherical top hat function

$$\sigma^2_{Sph}(R) = \int \frac{dk}{k}\left(\frac{k^3P(k)}{2\pi^2}\right)|W_k(R)|^2 = \int \frac{dk}{k}\Delta^2(k) \ \ 9\left(\frac{\sin kR - KR\cos kR}{(kR)^3}\right)^2 \tag{5.62}$$

One can easily find scaling between σ and R for a power law model.

$$P(k) = Ak^n \quad A \ \text{ is a normalization constant}$$

In this case

$$\sigma^2(R) = CR^{-(n+3)}; \quad C = \frac{9}{2\pi^2}\int y^{(n-2)}dy(\sin y - y\cos y)^2 \tag{5.63}$$

In general, it is more useful to write σ^2 as a function of $M = 4\pi R^3/3$.

$$\sigma^2(M) = \left\langle \left(\frac{\delta M}{M} \right)^2 \right\rangle \propto M^{\frac{-(n+3)}{3}} \tag{5.64}$$

Note that for scale invariant power spectrum $P(k) \propto k$

$$\Delta_\phi^2(k) = \frac{d\sigma^2}{dlnl} = constant$$

5.5.4 Moments of Counts-in-cells

When we make a discrete sampling of a continiuos field then it introduces a noise term in power spectrum $P(k)$ and two point correlation function $\xi(r)$. In order to see this effect let us consider a random distribution of particles with average density $\bar{n}$ and divide it into *cells* of various size. We can further subdivide each shell into *sub-cells* of so small size that a typical sub-shell either can have zero particles of one particle i.e., $n_i = 0, 1$.

Note that here we will use:

$$< n_i >=< n_i^2 >=< n_i^3 >= ... \tag{5.65}$$

Total number N of particles in a cell

$$N = \sum_i n_i \tag{5.66}$$

Now we can take the various moments of this quantity

$$< N >= \sum_i < n_i >= \bar{n}V \tag{5.67}$$

Here V is the volume of a cell.

$$< N^2 >= \sum_i < n_i^2 > + \sum_{i \neq j} < n_i n_j > \tag{5.68}$$

Now using the relation

$$< n_i n_j >= \bar{n}^2 \delta V_i \delta V_j (1 + \xi_2(i, j)) \quad \text{and} \quad < n_i^2 >=< n_i >$$

$$< N^2 >= \sum_i < n_i > + \bar{n}^2 \sum_{i \neq j} \delta V_i \delta V_j (1 + \xi_2(i, j))$$

or

$$< N^2 >= \bar{n}V + (\bar{n}V)^2 + \bar{n}^2 \sum_{i \neq j} \delta V_i \delta V_j \xi_2(i, j)$$

or

$$< N^2 >= \bar{n}V + < N >^2 + \bar{n}^2 \sum_{i \neq j} \delta V_i \delta V_j \xi_2(i, j)$$

or

$$< N^2 > - < N >^2 = \bar{n}V + \bar{n}^2 \sum_{i \neq j} \delta V_i \delta V_j \xi_2(i,j) \tag{5.69}$$

Note that even in the case where are no correlations i.e., $\xi_2 = 0$, there are fluctuation in the number of objects in the shells of a given size due to the first term of right hand side. These fluctuations are called the *shot noise*. The shot noise increases with the size of the shells and it is caused by the self correlations of particles.

Exercise:

If we define

$$\delta(r_i) = \frac{n(r_i) - \bar{n}_i}{\bar{n}_i}$$

then show that

$$< \delta(r_i)\delta(r_j) >_{shot} = \frac{\delta_D(r_{ij})}{\bar{n}_i}$$

Solution:

$$
\begin{aligned}
< \delta(r_i)\delta(r_j) > &= \left\langle \left(\frac{n(r_i) - \bar{n}_i}{\bar{n}_i}\right)\left(\frac{n(r_j) - \bar{n}_j}{\bar{n}_j}\right) \right\rangle \\
&= \frac{< n(r_i)n(r_j) >}{\bar{n}_i \bar{n}_j} - 1 \\
&= \frac{< (n(r_i))^2 >}{(\bar{n}_i)^2} + \frac{< n(r_i)n(r_j) >_{i \neq j}}{\bar{n}_i \bar{n}_j} - 1 \\
&= \frac{< n(r_i) >}{(\bar{n}_i)^2} + 1 + \xi_2(r_i, r_j)_{i \neq j} - 1 = \frac{\delta_R(r_i, r_j)}{\bar{n}_i} + \xi_2(r_i, r_j)_{i \neq j}
\end{aligned}
$$

Since

$$< (n(r_i))^2 > = < n(r_i) > = \bar{n}_i \quad \text{and} \quad < n(r_i)n(r_j) > = \bar{n}_i \bar{n}_j (1 + \xi_2(r_i, r_j))$$

Moments of the counts-in-cells are defines as by the following relation:

$$\mu_n = < (N - < N >)^n >$$

from the equation (5.69)

$$\mu_2 = < (N - < N >)^2 > = < N^2 > - < N >^2 = < N > + \frac{< N >^2}{V^2} \int \int \xi_2(1,2) dV_1 dV_2$$

and μ_3 turns out as the following

$$m_3 = 3 < (N - < N >)^2 > -2 < N > + \frac{< N >^3}{V^3} \int \int \xi_3(1,2,3) dV_1 dV_2 dV_3$$

Now if underlying fluctuations are **Gausaaian** and galaxies are distributed as **random Poisson point process** then the distribution function f_N will be the following:

$$f(N) = \frac{1}{(2\pi < N >^2 \sigma^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{e^{-\lambda}\lambda^n}{N!} exp\left[\frac{-(\lambda - \bar{\lambda})}{2 < N > \sigma^2}\right] d\lambda \tag{5.70}$$

Note that in this case

$$\xi_N = 0 \quad \text{for} \quad N > 2$$

and

$$\mu_2 = \sigma^2 < N >^2 + < N >$$

$$\mu_3 = 3\mu_2 - 2 < N >$$

$$\mu_4 = 7\mu_2 + 3\mu_2^2 - 6 < N > \tag{5.71}$$

5.5.5 Selection function

In any galaxy survey we cannot observe galaxies which are fainter than the *limiting magnitude* of the survey. So in order to take this into account we need to define a function $p(x)$ called the *selection function*. This function gives the probability that a galaxy at a distance x is included in the catalouge.

The probability density that a galaxy of luminosity L_i with redshift z_i is selected is

$$p_i \propto \frac{\phi(L_i)}{\int_{L_{min}}^{\infty} \phi(L)dL} \tag{5.72}$$

Here $\phi(L_i)$ is called the *Luminosity function* which is defined in term of the number density of galaxies which have luminosity in the range $[L, L + dL]$.

$$dn(L) = \phi(L)dL \tag{5.73}$$

If we integrate this function over the full range of luminosities then we get the number density of the galaxies.

$$n = \int \phi(L)dL \tag{5.74}$$

It has been found that the following luminosity function which is called the *Schechter luminosity function* matches the observational data up to certain extent.

$$\phi(L)dL = \phi^\star \left(\frac{L}{L^\star}\right)^\alpha e^{\frac{-L}{L^\star}} d\left(\frac{L}{L^\star}\right) \tag{5.75}$$

Now we can compute the number density of objects from this luminosity function.

$$n = \phi^\star \Gamma(\alpha + 1) \tag{5.76}$$

In order to keep n finite, we need $\alpha > -1$.

From the above luminosity function we can also find luminosity.

$$< L >= \int L\phi(L)dL = \phi^\star L^\star \Gamma(\alpha + 2) \tag{5.77}$$

Now again in order to keep $< L >$ finite we need $\alpha > -2$.

There are three independent parameters in the Schechter luminosity function: $\phi^\star$ fixes the normalization, α gives the slope at the fainter end and $L^\star$ gives the characteristic luminosity. In order to fix these parameters we use the method of *maximum liklihood*.

Liklihood function $\mathcal{L}$ is defined as follows:

$$\mathcal{L} = \Pi_i \quad p_i \tag{5.78}$$

Where p_i is defined by equation (5.72). Note that this method cannot be used to fix the $\phi^\star$ because equation (5.72) is independent from the normalization.

5.6 Cosmological perturbations

It has been shown by many observations that the dominant component of the universe is non-baryonic dark matter. Most of the models of structure formation assume that structure in the universe are formed by gravitational enhancement of primordial inhomogeneities (perturbations) of dark matter. As long as density perturbations are small, various Fourier modes of perturbations evolve independently and their evolution can be studied by linear perturbation theory, but when these become large mode coupling becomes important and linear perturbation theory no longer remains valid. Galaxies, clusters of galaxies and other structures in the universe correspond to large perturbations.

The main components that contribute in the energy budget of the universe are: dark matter, radiation and cosmological constant. If we perturb density of any one of the components then the evolution of this perturbation is enhanced by the perturbed potential and suppressed by the cosmic expansion. Let us consider a case of vanishing cosmological constant and, let there is only dark matter and radiation, so the energy density perturbation

$$\delta = \frac{\rho_R \delta_R + \rho_{DM} \delta_{DM}}{\delta_R + \delta_{DM}} = \frac{\delta_R + y\delta_{DM}}{1+y} \tag{5.79}$$

where $y = \frac{\rho_{DM}}{\rho_R}$ For a two components system we need one more variable; let us take it entropy per dark matter particle, so

$$s \propto \frac{T_R^3}{\rho_{DM}} \tag{5.80}$$

and so

$$\sigma = \frac{\delta s}{s} = 3\left(\frac{\delta T_R}{T_R}\right) - \frac{\delta \rho_{DM}}{\rho_{DM}} = \frac{3}{4}\delta_R - \delta_{DM} \tag{5.81}$$

here we have used $\rho_R \propto T_R^4$

Now there are two interesting case

- $\sigma = 0, \delta \neq 0$:- Such perturbations are called the **isentropic perturbation** ; for this case $3\delta_R = 4\delta_{DM}$.

- $\sigma \neq 0, \delta = 0$:- Such perturbations are called **iso-curvature perturbations** .

5.7 Linear perturbation theory

On the basis either the wavelength of the cosmolical perturbation is smaller or larger than the hubble scale, they are said be sub-horizon or super-horizon. In principle, we must use general relativity for the later case, however, if the perturbations are spherical then even without that we can find their growth for some simple cases.

5.7.1 Super-horizon scale ($\lambda \geq d_H$)

Let us consider a spherical over-dense region in an otherwise uniform background. Now in this case the matter inside the region will not be affected by the matter outside gravitationally. We can consider the inside over-dense region like a $k > 0$ universe and the outside region like a $k = 0$ universe. Now we can write Friedman equations for the outside and inside region.

$$H^2 = \frac{8\pi G\rho_b}{3}$$

and

$$H^2 + \frac{k}{a^2} = \frac{8\pi G\rho}{3}$$

from the above two equations:

$$\delta = \frac{\rho - \rho_b}{\rho_b} \propto \frac{1}{a^2\rho_b}$$

From this expression it is clear that

$$\begin{aligned} \delta &\propto a \quad \text{For matter dominated universe} \\ &\propto a^2 \quad \text{For radiation dominated universe} \end{aligned}$$

5.7.2 Sub-horizon scales ($\lambda \leq d_H$)

There are two ways by called the Eulerian and Lagrangian which we can use to describe the motion of a fluid. In the former case partial derivatives are calculated in a coordinate system which does not move with the laboratory frame, and in the later the derivates are computed in a coordinate system which follows the particles motion.

Eulerian: Langrangian:

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\mathbf{v}) = 0 \qquad \frac{d\rho}{dt} = -\rho\nabla.\,\mathbf{v} \tag{5.82}$$

$$\frac{\partial\,\mathbf{v}}{\partial t} + (\,\mathbf{v}.\nabla)\,\mathbf{v} = -\frac{\nabla P}{\rho} - \nabla\Phi \qquad \frac{d\,\mathbf{v}}{dt} = -\frac{\nabla P}{\rho} - \nabla\Phi \tag{5.83}$$

$$\nabla^2\Phi = 4\pi G\rho \qquad \nabla^2\Phi = 4\pi G\rho \tag{5.84}$$

Now we can perturb the various physical quantities in the following way:

$$\rho \longrightarrow \rho_0 + \delta\rho \;\; \text{or} \;\; \rho = \rho_0\left(1 + \frac{\delta\rho}{\rho}\right) = \rho_0(1+\delta)$$

Here δ, is the density contrast which is the main quantity of interest.

$$P \longrightarrow P_0 + \delta P$$

$$\mathbf{v} \longrightarrow \mathbf{v}_0 + \delta\mathbf{v}$$

$$\Phi \longrightarrow \Phi_0 + \delta\phi = \phi_0 + \delta\phi$$

Now we can write the unperturbed equations:

$$\frac{d\rho_0}{dt} = -\rho_0(\nabla .\ \mathbf{v_0})$$

$$\frac{d\ \mathbf{v_0}}{dt} = -\frac{\nabla P}{\rho_0} - \nabla\phi_0$$

$$\nabla^2\phi_0 = 4\pi G\rho_0$$

And the perturbed equations are as follows:

$$\frac{d(\rho_0 + \delta\rho)}{dt} = -(\rho_0 + \delta\rho)\nabla(\ \mathbf{v_0} + \delta\mathbf{v})$$

$$\frac{d(\ \mathbf{v_0} + \delta\mathbf{v})}{dt} = -\frac{\nabla(P_0 + \delta P)}{\rho_0 + \delta\rho} - \nabla(\phi_0 + \delta\phi)$$

$$\nabla^2(\phi_0 + \delta\phi) = 4\pi G(\phi_0 + \delta\phi)$$

Now subtracting the respective unperturbed equations from the perturbed equations and keeping only the terms linear in $\delta\rho$ and $\delta\ \mathbf{v}$ and taking the spatial distribution of $\mathbf{v_0}$ and P are uniform i.e., $\nabla\ \mathbf{v_0} = \nabla P_0 = 0$.

From the continuity equation:

$$\frac{d\delta\rho}{dt} = -\rho_0\nabla(\delta\mathbf{v})$$

or

$$\frac{d\delta}{dt} = -\nabla.\delta(\mathbf{v}) \tag{5.85}$$

From the Euler's equation:

$$\begin{aligned}
LHS &= \frac{d(\ \mathbf{v_0} + \delta\ \mathbf{v})}{dt} - \frac{d\ \mathbf{v_0}}{dt} \\
&= \frac{\partial(\ \mathbf{v_0} + \delta \mathbf{v})}{\partial t} + [(\ \mathbf{v_0} + \delta\ \mathbf{v}).\nabla](\ \mathbf{v_0} + \delta\ \mathbf{v}) - \frac{\partial\ \mathbf{v_0}}{\partial t} - (\ \mathbf{v_0}.\nabla)(\ \mathbf{v_0}) \\
&= \frac{d\delta\ \mathbf{v}}{dt} + \delta\ \mathbf{v}.\nabla\ \mathbf{v_0}
\end{aligned}$$

Now we can compute

$$\begin{aligned}
RHS &= -\frac{\nabla(P_0 + \delta P)}{(\rho_0 + \delta\rho)} - \nabla(\phi_0 + \delta\phi) + \frac{\nabla(P_0)}{\rho_0} + \nabla(\phi_0) \\
&\approx -\frac{\nabla\delta P}{\rho_0} - \nabla\delta\phi
\end{aligned}$$

Equating LHS and RHS

$$\frac{d\delta\ \mathbf{v}}{dt} + (\delta\ \mathbf{v}.\nabla)\ \mathbf{v_0} = -\frac{\nabla\delta P}{\rho_0} - \nabla(\delta\phi) \tag{5.86}$$

We can simply the second term of RHS by the following way:

$$[(\delta\ \mathbf{v}.\nabla)\ \mathbf{v_0}]_j = \delta v_i \nabla_i [v_0]_j = \delta v_i H \delta_{ij} = H\delta v_j$$

This means that

$$(\delta\ \mathbf{v}.\nabla)\ \mathbf{v_0} = H\delta\ \mathbf{v}$$

Now the Euler's equation can be written as:

$$\frac{d\delta\ \mathbf{v}}{dt} + H\delta\ \mathbf{v} = -\frac{\nabla\delta P}{\rho_0} - \nabla(\delta\phi) \tag{5.87}$$

Possion equation can be easily written as:

$$\nabla^2\delta\phi = 4\pi G\rho_0\delta$$

Now we have a set of the following three equations in comoving coordinates i.e., $x(t) = r(t)/a(t)$.

$$\frac{d\delta}{dt} = -\frac{1}{a}(\nabla_x.\delta\ \mathbf{v})$$

$$\frac{d\delta\ \mathbf{v}}{dt} + H\delta\ \mathbf{v} = -\frac{\nabla\delta P}{a\rho_0} - \frac{1}{a}\nabla(\delta\phi)$$

$$\nabla^2\delta\phi = 4\pi G\rho_0 a^2\delta$$

Now eliminating $\delta\ \mathbf{v}$ from the first two equations:

$$\frac{d^2\delta}{dt^2} + 2H\frac{d\delta}{dt} = \frac{\nabla^2\delta P}{\rho_0} + \nabla^2(\delta\phi) \tag{5.88}$$

Using the relations $\delta P = c_s^2\delta\rho$ and $\nabla^2\delta\phi = 4\pi G\rho_0\delta$

$$\frac{d^2\delta}{dt^2} + 2H\frac{d\delta}{dt} = c_s^2\nabla^2\delta + 4\pi G\rho_0\delta \tag{5.89}$$

Let consider the case when we can neglect the pressure (long wavelength):

$$\frac{d^2\delta(x,t)}{dt^2} + 2H\frac{d\delta(x,t)}{dt} = 4\pi G\rho_0\delta(x,t)$$

This is a second order differential equation which has two solutions which are called the growing and decaying solution. Let us write these in the following way.

$$\delta_1(x,t) = D_+(t)\delta(x,0) \text{ growing solution}$$

$$\delta_2(x,t) = D_-(t)\delta(x,0) \text{ decaying solution}$$

So we can write the equation in the following form also:

$$\frac{d^2D_\pm}{dt^2} + 2H\frac{dD_\pm}{dt} = 4\pi G\rho_0 D_\pm = \frac{3}{2}H_0^2\Omega_{NR}D_\pm \tag{5.90}$$

On the interesting things is that the hubble constant H also satisfy the above equations so we can identify it with the decaying solution. This is can shown by the following way:

Let us write the continuity equation:

$$\dot{\rho}_b + \nabla(\rho_b v) = 0$$

Here $v = Hx$ is the hubble velocity. So from the above equations

$$\dot{\rho}_b = -3H\rho_b$$

From the Friedman's equations:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G\rho}{3}$$

or

$$\ddot{H} + 2H\dot{H} = -\frac{4\pi G}{3}\dot{\rho}_b = 4\pi G\rho_b H = \frac{3H_0^2}{2}\Omega_{NR}H$$

This equation is identical to equation (5.90) so from this it clear that the hubble parameter H is also a solution.

From the above discussion it is clear that the hubble parameter H is a decaying solution δ_d we can also compute the growing solution by the following way:

Wronkison is

$$W = \dot{\delta}_g\delta_d - \dot{\delta}_d\delta_g = a^{-2}$$

so

$$\delta_g = \delta_d \int \frac{dt}{a^2 \delta_d^2} = \int \frac{dt}{a^2 H^2(t)} = H(a) \int \frac{da}{(Ha)^3} \tag{5.91}$$

or

$$\delta_g = \frac{5}{2} \Omega_{NR} H(a) \int_0^a \frac{dx}{x^3 H^3(x)} \tag{5.92}$$

Where

$$H(x) = \left[\Omega_{NR} x^{-3} + \Omega_V + (1 - \Omega_{NR} - \Omega_V) x^{-2} \right]^{1/2}$$

Solving a differential equation using Wronskian

If we have two differential functions $y_1(x)$ and $y_2(x)$ then their Wronskian is defined as:

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

or

$$\left(\frac{y_2}{y_1} \right)' = \frac{W(y_1, y_2)}{y_1^2}$$

If $y_1(x)$ and $y_2(x)$ are the solutions of the differential equation.

$$y'' + P(x) y' + Q(x) y = 0$$

then

$$W(y_1, y_2)(x) = W(y_1, y_2)(x_0) \exp \left(- \int_0^x P(t) dt \right)$$

5.7.3 Jeans length

In oder to see the effect of pressure let us decompose the density field δ in fourier components.

$$\delta(x, t) = \sum_0^\infty \delta_k e^{-i \ \mathbf{k.x}}$$

Now equation (5.89) can be written as:

$$\frac{d^2 \delta_k}{dt^2} + 2H \frac{d\delta_k}{dt} - 4\pi G \rho_0 \delta_k \left(1 - \frac{c_s^2 k^2}{4\pi G \rho_0 a^2} \right) = 0$$

or

$$\frac{d^2\delta_k}{dt^2} + 2H\frac{d\delta_k}{dt} - 4\pi G\rho_0\delta_k\left(1 - \frac{k^2}{k_J^2}\right) = 0 \tag{5.93}$$

or

$$\frac{d^2\delta_k}{dt^2} + 2H\frac{d\delta_k}{dt} - \omega^2\delta_k = 0 \quad \text{where} \quad \omega^2 = 4\pi G\rho_0\left(1 - \frac{k^2}{k_J^2}\right)$$

where

$$k_J^2 = \frac{4\pi G\rho_0 a^2}{c_s^2} \quad \text{and} \quad L_J = \frac{2\pi a}{k_J} = \frac{c_s}{\sqrt{4\pi G\rho_0}} \tag{5.94}$$

Here k_J and L_J are called the *Jeans wavenumber* and *Jeans length* respectively. On the basis of either $k > k_J$ or $k < k_J$ equation (5.93) have different solutions.

- $k > k_J$:- In this case $\omega^2 < 0$ so we have the following type of oscillatory solutions:

$$\delta_k(t) = Ae^{i\omega t} + Ae^{-i\omega t} \tag{5.95}$$

  This simply means that perturbations smaller than the jeans length i.e., $k > k_J$ can not grow.

- $k < k_J$:- In this case $\omega^2 > 0$ and so we have the following type of growing and decaying solutions:

$$\delta_k(t) = Ce^{\omega t} + De^{\omega t} \tag{5.96}$$

Note that the equation (5.93) is like the equation of a damped harmonic oscillator so our coefficients A, B, C and D all are time dependent.

5.7.4 Some special case

Let us consider first the case when $k << k_J$ or the scales of interest are very large in comparison in comparison to the jeans scale. In the case we can drop the pressure term from our equation.

$$\frac{d^2\delta_k}{dt^2} + 2H\frac{d\delta_k}{dt} - 4\pi G\rho_0\delta_k = 0 \tag{5.97}$$

Now let us consider the following two simple cases:

- **Matter dominated flat universe:**

  In this case $a(t) \propto a^{2/3}$ so

$$\frac{d^2\delta_k}{dt^2} + \frac{4}{3t}\frac{d\delta_k}{dt} - 4\pi G\rho_0\delta_k = 0$$

  since

$$H^2 = \frac{8\pi G\rho_0}{3} = \frac{4}{9t^2} \implies 4\pi G\rho_0 = \frac{2}{3t^2}$$

so

$$\frac{d^2\delta_k}{dt^2} + \frac{4}{3t}\frac{d\delta_k}{dt} - \frac{2}{3t^2}\delta_k = 0$$

Now we can find its power law solutions i.e., $\delta_k \propto t^m$

$$m(m-1) + \frac{4}{3}m - \frac{2}{3} = 0$$

or

$$m^2 + \frac{1}{3}m - \frac{2}{3} = (m - 2/3)(m+1) = 0$$

This means that in this case the solutions are:

$$\delta_k(t) = At^{2/3} + Bt^{-1} \quad \text{or} \quad Aa + Ba^{-3/2} \tag{5.98}$$

Here the first part is the growing and the second part ism the decaying solution.

- **Radiation dominated flat universe:**

  Since in this case pressure is not negilibale so we must use the following fluid equations:

$$\frac{d\rho}{dt} + \nabla.\left(\rho + \frac{P}{c^2}\right)\mathbf{v} = 0$$

$$\left(\rho + \frac{P}{c^2}\right)\left(\frac{\partial\ \mathbf{v}}{dt} + \ \mathbf{v}.\nabla.\mathbf{v}\right) = -\nabla P - \left(\rho + \frac{P}{c^2}\right)\nabla\phi = 0$$

and

$$\nabla^2\phi = 4\pi G\left(\rho + \frac{P}{c^2}\right)$$

perturbing these equations and simplifying we get:

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k - \frac{32\pi G\rho}{3}\delta_k = 0$$

Now for radiation dominated universe:

$$\frac{\dot{a}}{a} = \frac{1}{2t}; \quad \text{and} \quad \rho = \frac{1}{32\pi G t^2}$$

In this case the solutions turn out to be in the following form:

$$\delta_k(t) = At + Bt^{-1}$$

5.7.5 Growing mode as a time parameter

Let start with the following coordinate transformation between physical (r) and comoving coordinates:

$$r(t) = a(t)x(t); \quad \nabla_r = \frac{\nabla_x}{a}$$

so the equation of motion:

$$\ddot{r} = -\nabla_r \Phi$$

or in comoving coordinates:

$$a\ddot{x} + 2\dot{a}\dot{x} + \ddot{a}x = \frac{-\nabla_x \Phi}{a} = g_0 + g$$

or

$$a\ddot{x} + 2\dot{a}\dot{x} = -\frac{1}{a}\nabla_x(\Phi + \frac{1}{2}a\ddot{a}x^2)$$

or

$$\ddot{x} + 2\frac{\dot{a}}{a}\dot{x} = -\frac{\nabla\phi}{a^2}$$

where

$$\phi = \Phi + \frac{1}{2}a\ddot{a}x^2$$

Note that the Poisson equation is given by

$$\nabla_x^2\phi = 4\pi G a^2 \rho_b \delta$$

Now we use the going solution $\delta_+(t) = b(t)$ as a time variable in our equation of motion.

$$\frac{dx}{dt} = \dot{b}\frac{dx}{db} = \dot{b}w \quad \text{where} \quad w = \frac{dx}{db}$$

$$\frac{d^2x}{dt^2} = \dot{b}^2\frac{dw}{db} + \ddot{b}w$$

So now our equation of motion:

$$\dot{b}^2\frac{dw}{db} + \left(\ddot{b} + 2\frac{\dot{a}}{a}\dot{b}\right)w = -\frac{\nabla_x\phi}{a^2}$$

Now since b is the growing mode so it must satisfy the equation (5.97) or

$$\ddot{b} + 2\frac{\dot{a}}{a}\dot{b} = 4\pi G\rho_b b = \frac{3}{2}H_0^2\frac{b}{a^3}$$

So now our equation of motion becomes:

$$\frac{dw}{db} + \left(\frac{3H_0^2 b}{2\dot{b}^2 a^3}\right)w = -\frac{\nabla_x\phi}{\dot{b}^2 a^2}$$

$$\frac{dw}{db} = -\frac{3}{2b}\left(\frac{H_0^2 b^2}{\dot{b}^2 a^3}\right)\left[w + \nabla_x\left(\frac{2a}{3bH_0}\phi\right)\right] = -\frac{3}{2b}A(w + \nabla_x\psi)$$

where

$$A = \frac{H_0^2 b^2}{\dot{b}^2 a^3} = \frac{8\pi G\rho_0}{3}\left(\frac{\dot{a}}{a}\right)^2\left(\frac{\dot{a}b}{\dot{b}a}\right)^2\frac{1}{a^3} = \left(\frac{8\pi G\rho}{3H^2}\right)\left(\frac{\dot{a}b}{\dot{b}a}\right)^2 = \frac{\rho}{\rho_c}\left(\frac{\dot{a}b}{\dot{b}a}\right)^2$$

and

$$\psi = \frac{2}{3H_0^2}\left(\frac{a}{b}\right)\phi$$

Note that in this case our set of equation comes out be as follows:

$$w = \frac{dx}{db} \tag{5.99}$$

$$\frac{dw}{db} = -\frac{3}{2b}A(w + \nabla\psi) \tag{5.100}$$

$$\nabla^2\psi = \frac{\delta}{b} \tag{5.101}$$

5.7.6 Alternate method 1

For an ideal fluid the main equations in physical coordinate system are as follows:

$$\left(\frac{\partial\rho}{\partial t}\right)_r + \nabla_r.\rho\mathbf{u} = 0 \tag{5.102}$$

and

$$\left(\frac{\partial u}{\partial t}\right)_r + (\ \mathbf{u}.\nabla_r)u = -\frac{\nabla_r P}{\rho} - \nabla_r\phi \tag{5.103}$$

Now we can go from the physical coordinates to comoving coordinates:

$$r(t) = a(t)x(t) \quad \text{and} \quad \dot{r} = \dot{a}x + a\dot{x} = \dot{a}x + \mathbf{v}$$

and

$$\left(\frac{\partial}{\partial t}\right)_r = \left(\frac{\partial}{\partial t}\right)_x - \frac{\dot{a}}{a}(x.\nabla) \quad \text{and} \quad \nabla_r = \frac{\nabla_x}{a}$$

Now in the new coordinate system.

$$\frac{\partial\rho}{\partial t} + 3\frac{\dot{a}}{a}\rho + \frac{1}{a}\nabla.\rho\mathbf{v} = 0 \tag{5.104}$$

and

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{a}(\ \mathbf{v}.\nabla)\mathbf{v} + \frac{\dot{a}}{a} = -\frac{1}{\rho a}\nabla P - \frac{\nabla\phi}{a} \tag{5.105}$$

Now using $\rho = \rho_0(1+\delta)$ and keeping only the term linear in δ and $\mathbf{v}$:

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla.\rho\mathbf{v} = 0 \tag{5.106}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{\dot{a}}{a} = -\frac{1}{\rho a}\nabla P - \frac{\nabla\phi}{a} \tag{5.107}$$

Again we can ellinate $\mathbf{v}$ from the above two equation we get:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G\rho_b\delta$$

where we have used $P = constant$ and $\nabla^2\phi = 4\pi G\rho_b a^2\delta$

5.7.7 Peculiar velocity field

Conntinuty equation:

$$\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a}\nabla_x(\rho\mathbf{v}) = 0$$

or

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla_x[(1+\delta)\mathbf{v})] = 0$$

Keeping only the linear term:

$$\frac{\partial \delta}{\partial t} = -\frac{1}{a}\nabla_x\ \mathbf{v}$$

or

$$\nabla.\ \mathbf{v} = -a\dot{\delta}$$

Now let us use $\delta(x,t) = A(x)D(t)$ so:

$$\nabla.\ \mathbf{v} = -aA\dot{D} = -a\delta\frac{\dot{D}}{D} = -\delta\dot{a}\left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{D}}{D}\right) = -\delta A\dot{a}f$$

where

$$f = \frac{a}{\dot{a}}\frac{\dot{D}}{D} = \frac{1}{H}\frac{\dot{D}}{D}$$

Now our equation is

$$\nabla.\ \mathbf{v} = -faH\delta$$

It's solution is:

$$\mathbf{v}(x) = \frac{afH}{4\pi} \int \frac{y-x}{|y-x|^3} \delta(y) d^3y \tag{5.108}$$

Now we can relate this equation to the Possion equation.

$$\nabla^2 \phi = 4\pi G \rho_b a^2 \delta$$

or

$$\nabla \left(\frac{\nabla \phi}{a} \right) = 4\pi G \rho_b a \delta$$

or

$$\nabla \, \mathbf{g} = 4\pi G \rho_b a \delta \ \text{ where } \ g = \frac{\nabla \phi}{a}$$

and its solution is

$$\mathbf{g}(x) = G \rho_b a \int \frac{y-x}{|y-x|^3} \delta(y) d^3y \tag{5.109}$$

from equation (5.108) and (5.109)

$$\mathbf{v} = \frac{2f}{3\Omega H} \, \mathbf{g} \tag{5.110}$$

Note that for a spherical symmetric case:

$$\mathbf{v}(x) = -\frac{afH}{x^2} \int_0^x y^2 dy \delta(y) = -\frac{1}{3} f H a \, \mathbf{x} \delta(x)$$

$$\delta = -\frac{1}{aHf} (\nabla_x . \, \mathbf{v}) = 0 \ \text{ where } \ f = \frac{dlog\delta}{dloga} \approx \Omega_0^{0.6} + \frac{\Omega_\Lambda}{70} \left(1 + \frac{\Omega_0}{2} \right)$$

5.8 Non linear gravitational clustering

5.8.1 Mode coupling

In order to study the statistics of cosmological perturbations in dark matter, dark matter is considered in terms of collisionless particles. Which interact with each other only gravitationally.

$$\rho(x,t) = \frac{1}{a^3(t)} \sum_i m_i \delta_D^3 (x - x_i)$$

Here δ_D is the usual Dirac delta.

Now we can compute average or background density also.

$$\rho_b(t) = \frac{1}{V} \int d^3x \rho(x,t) = \frac{M}{a^3 V} = \frac{\rho_0}{a^3}$$

and so the density contrast $\delta(x,t)$:

$$\delta(x,t) = \frac{\rho(x,t)}{\rho_b(t)} - 1 = \frac{V}{M}\sum\nolimits_i m_i \delta_D^3(x - x_i) - 1$$

This can be written in fourier space.

$$\delta_k(t) = \frac{1}{V}\int d^3x \delta(x,t) e^{ik.x} = \frac{1}{M}\sum\nolimits_i m_i e^{ik.x_i} - \delta_D^3(k)$$

considering only the case when $k \neq 0$.

$$\delta_k(t) = \frac{1}{M}\sum\nolimits_i m_i e^{i\mathbf{k}.\mathbf{x_i(t)}} = \frac{1}{N}\sum\nolimits_i e^{i\mathbf{k}.\mathbf{x_i(t)}} \tag{5.111}$$

From this equation we can find:

$$\frac{\partial^2 \delta_k}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta_k}{\partial t} = \frac{1}{N}\sum \left[i\mathbf{k}.\left(\ddot{x}_i + 2\frac{\dot{a}}{a}\dot{x}\right) - (i\mathbf{k}.\dot{\mathbf{x}}_\mathbf{i}(\mathbf{t}))^2 \right] e^{i\mathbf{k}.\mathbf{x_i}(t)}$$

using:

$$\ddot{x}_i + 2\frac{\dot{a}}{a}\dot{x} = -\frac{\nabla_x \phi}{a^2}$$

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k = \frac{1}{N}\sum_i \left[i\mathbf{k}.\left(\frac{-\nabla\phi(x_i)}{a^2}\right) - (i\mathbf{k}.\dot{\mathbf{x}}_\mathbf{i})^2 \right] e^{i\mathbf{k}.\mathbf{x_i}(t)} = C_K - B_K \tag{5.112}$$

Where

$$C_K = \frac{1}{N}\sum_i i\mathbf{k}.\left(\frac{-\nabla\phi(x_i)}{a^2}\right)$$

and

$$B_K = \frac{1}{N}\sum_i (i\mathbf{k}.\dot{\mathbf{x}}_\mathbf{i}(\mathbf{t}))^2 e^{i\mathbf{k}.\mathbf{x_i}}$$

In order to find C_K we need to solve the Possion equation.

$$\nabla_x^2 \phi = 4\pi G a^2 \rho_b \delta$$

or

$$-{k'}^2 \phi_{k'} = 4\pi G a^2 \rho_b \delta_{k'}$$

or

$$\phi_{k'} = 4\pi G a^2 \rho_b \frac{\delta_{k'}}{-{k'}^2} = 4\pi G a^2 \rho_b \frac{1}{-{k'}^2}\frac{1}{N}\sum_j e^{ik'.x_j}$$

so

$$\phi(x_i) = \sum_p \phi_{k'} e^{-i\mathbf{k'}.\mathbf{x_i(t)}} = \frac{4\pi G\rho_b a^2}{N}\sum_{k'}\left[\frac{1}{-{k'}^2}\sum_j e^{i(\mathbf{x_j}-\mathbf{x_i}).\mathbf{k'}}\right]$$

or

$$\nabla\phi(x_i) = \frac{4\pi G\rho_b a^2}{N}\sum\nolimits_{k'}(-i\ \mathbf{k'})\left[\frac{1}{-k'^2}\sum_j e^{i(\mathbf{x_j}-\mathbf{x_i}).\mathbf{k'}}\right]$$

or

$$C_K = \frac{1}{N}\sum_i\left[i\mathbf{k}\left(\frac{4\pi G\rho_b}{N}\sum_{k'}(-i\ \mathbf{k'})\left[\frac{1}{-k'^2}\sum_j e^{i(\mathbf{x_j}-\mathbf{x_i}).\mathbf{k'}}\right]\right)e^{i\mathbf{k}.\mathbf{x_i}(\mathbf{t})}\right] \tag{5.113}$$

$$= \left[\sum_{k'}\frac{\mathbf{k}.\mathbf{p}}{k'^2}\left(\sum\nolimits_i e^{i(\mathbf{k}-\mathbf{k'}).\mathbf{x_i})}\right)\left(\sum_j e^{i(\mathbf{k'}.\mathbf{x_j})}\right)\right] \tag{5.114}$$

$$= 4\pi G\rho_b\sum_{k'}\frac{\mathbf{k}.\mathbf{k'}}{k'^2}\delta_{k'}\delta_{k-k'} \tag{5.115}$$

$$= 4\pi G\rho_b\delta_k + 4\pi G\rho_b\sum_{k'\neq k}\frac{\mathbf{k}.\mathbf{k'}}{k'^2}\delta_{k'}\delta_{k-k'} = 4\pi G\rho_b\delta_k + A_k \tag{5.116}$$

Now equation (5.112) can be written as:

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k = 4\pi G\rho_b\delta_k + A_k - B_k \tag{5.117}$$

where

$$A_k = 4\pi G\rho_b\sum_{k'\neq k}\frac{\mathbf{k}.\mathbf{k'}}{k'^2}\delta_{k'}\delta_{k-k'}$$

Argument $\delta_{k'}\delta_{k-k'}$ is invariant under the transformation $k'' = k - k'$ so we can write:

$$A_k = 4\pi G\rho_b\sum_{k'\neq k}\frac{\mathbf{k}.\mathbf{k'}}{k'^2}\delta_{k'}\delta_{k-k'} \tag{5.118}$$

$$= 4\pi G\rho_b\sum_{k'\neq k}\left[\frac{\mathbf{k}.\mathbf{k'}}{k'^2} + \frac{\mathbf{k}.(\mathbf{k}-\mathbf{k'})}{|k-k'|^2}\right] \tag{5.119}$$

since $4\pi G\rho_b = \frac{3}{2}H^2\Omega_{NR}$ so equation (5.117) can be written in the following form also.

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k = \left(\frac{3H^2\Omega_{NR}}{2}\right)\delta_k + A_k - B_k \tag{5.120}$$

This is a second order nonlinear partial differential equation. It show that the growth of perturbations at some scale k depends on the perturbations at all other scales k' or this equations shows mode coupling between various modes.

In the above equation A_k and B_k are the coupling terms. In the case when these are smalls or $A_k - B_k$ is small, this equation can be solved exactly and gives the results of linear perturbation theory.

5.9 Spherical Collapse

Let us consider a homogeneous sphere made from concentric shells at a certain time t the acceleration of a shell of radius R will be

$$\ddot{R} = -\frac{GM}{R^2}$$

We can integrate this equation.

$$\frac{1}{2}\dot{R}^2 - \frac{GM}{R} = E$$

or

$$\frac{1}{2}\dot{R}^2 = \frac{GM}{R} + E$$

Here E is a constant, which can be identified with the total energy. Let us consider the case when E is negative i.e., $E = -|E|$. In this case:

$$t = \frac{1}{\sqrt{2|E|}} \int \frac{\sqrt{R}}{\sqrt{\frac{GM}{|E|} - R}}$$

Now let us make the following substitution:

$$R = \frac{GM}{|E|} \sin^2 \frac{\theta}{2} = \frac{GM}{2|E|}(1 - \cos\theta) = A(1 - \cos\theta) \tag{5.121}$$

This gives us:

$$t = \frac{1}{\sqrt{2|E|}} \frac{GM}{2|E|}(\theta - \sin\theta) = B(\theta - \sin\theta) \tag{5.122}$$

Note that $A^2 = 2|E|B^2$.

These equations show that for $E < 0$, the spherical shell firstly begins expand slower than the cosmic expansion and then at certain radius $R = R_m$ it stops expanding; this stage is called the *turn-around*. At the turn-around all energy of the shell is stored as potential energy. After reaching the turn-around, the shell start to collapsing. In an ideal situation it should keep collapsing till it does not become a black hole, but in a real collapse, it virialise $2K + U = 0$ before that.

One can write the above solutions in the following form also:

$$\frac{R}{R_m} = \frac{1}{2}(1 - \cos\theta) \;\; \text{and} \;\; \frac{t}{t_m} = \frac{1}{\pi}(\theta - \sin\theta)$$

One can find also find the variation of background density from the above equations:

$$t = \frac{1}{\sqrt{2|E|}} \frac{GM}{2|E|}(\theta - \sin\theta) \approx \frac{1}{\sqrt{2|E|}} \frac{GM}{2|E|} \frac{\theta^3}{6}$$

or

$$\theta = \left(\frac{6(2|E|)^{3/2}}{GM}\right)^{1/3} t^{1/3}$$

now

$$R = \frac{GM}{2|E|}(1-\cos\theta) \approx \frac{GM}{2|E|}\frac{\theta^2}{2}$$

now substituting the value of θ:

$$R = \frac{GM}{2|E|}(1-\cos\theta) \approx \frac{GM}{2|E|}\frac{1}{2}\left(\frac{6}{GM}\right)^{2/3} 2|E|t^{2/3}$$

or

$$R = \left(\frac{9GM}{2}\right)^{1/3} t^{2/3}$$

This is nothing but Einstein-de Sitter solution i.e., $R \propto t^{2/3}$.
We can obtain the variation of the background density by the following way:

$$R = \left(\frac{9Gt^2}{2}\right)^{1/3} M^{1/3} = \left(\frac{3}{4\pi\rho_b}\right)^{1/3} M^{1/3}$$

so

$$\rho_b(t) = \frac{1}{6\pi G t^2}$$

Let us compute the linear and nonlinear density contrast at turn-around and virialization.

5.9.1 Density contrast

We can compute:

$$1+\delta(\theta) = \frac{\rho(\theta)}{\rho_b(\theta)} = \frac{\frac{3M}{4\pi R^3}}{\frac{1}{6\pi G t^2}} = \frac{9}{2}MG\frac{t^2}{R^3} = \frac{9}{2}MG\frac{B^2(\theta-\sin\theta)^2}{A^3(1-\cos\theta)^3}$$

Now using the relation $A^2 = 2|E|B^2$ and $A = GM/2|E|$:

$$1+\delta_{NL}(\theta) = \frac{9}{2}\frac{(\theta-\sin\theta)^2}{(1-\cos\theta)^3} \tag{5.123}$$

This is the nonlinear estimate of density contrast which is exact. We can find the linear estimate from the above expression also.

$$\begin{aligned} 1+\delta_L(\theta) &= \frac{9}{2}\left[1-\left(\theta-\frac{\theta^3}{6}+\frac{\theta^5}{120}-...\right)^2\right]\left[1-\left(1-\frac{\theta^2}{2}+\frac{\theta^4}{24}-...\right)^{-3}\right] \\ &\approx 1+\frac{3}{20}\theta^2 \end{aligned}$$

so we can write:

$$\delta_L(\theta) = \frac{3}{20}\theta^2 \tag{5.124}$$

note that:

$$R_L(\theta) = \left(\frac{R_m}{4}\right)\theta^2 \quad \text{and} \quad t_L(\theta) = \left(\frac{t_m}{6\pi}\right)\theta^3$$

So from the above expression it is clear that $\delta_L(R) \propto R$, which is what we expect.

Let us find linear and linear values of density contrast at turn around and virialization. i.e., $\theta = \pi$:

1. **Turn around:**

   Since at turn around $\theta = \pi$ so

$$1 + \delta_{NL}(\pi) = \frac{9\pi^2}{16} = 5.55 \quad \text{or} \quad \delta_{NL} = 4.55 \tag{5.125}$$

$$1 + \delta_L(\pi) = \frac{3\pi^2}{20} = 1.47 \tag{5.126}$$

2. **Virialization:**

$$1 + \delta_{NL}(\theta) = \frac{9}{2}\frac{(\theta - \sin\theta)^2}{(1 - \cos\theta)^3}$$

   Virial condition is give by

$$2K + W = 0; E = W - \frac{W}{2} = \frac{W}{2} \tag{5.127}$$

   where K and W are the kinetic and potential energies respectvely.

$$E_V = -\frac{GM}{2R_v}$$

   but we know that $E_t = -\frac{GM}{R_m}$ and $E_v = E_t$ so

$$R_v = \frac{R_m}{2} \tag{5.128}$$

   This equations shows that the virial radius is half of the turn-around or maximum radius; so the density at virialization will be 8 times high than the density at turnaround. From the simple consideration we know

$$\frac{R_v}{R_m} = \frac{1}{2}$$

and $R \propto t^{\frac{2}{3}}$ so

$$\frac{t_v}{t_m} = \left(\frac{1}{2}\right)^{\frac{2}{3}}$$

and since background density falls as $\rho_b \propto \frac{1}{t^2}$ therefore the density at virialization is less dense by a factor of 4 and

$$1 + \delta_{NL} = \frac{9\pi^2}{16}\left(\frac{R_m}{R_v}\right)^3\left(\frac{\bar{\rho}_m}{\bar{\rho}_v}\right) = \frac{9\pi^2}{16} \times 8 \times 4 \approx 178 \tag{5.129}$$

Now we can compute the density contrast also.

$$\delta_L(t) = \frac{\rho}{\bar{\rho}} - 1 = \frac{3}{20}\theta^2 = \frac{3}{20}\left(\frac{6\pi t}{t_m}\right)^{2/3}$$

At virialization $t = 2t_m$ we find

$$\delta_{linear} = (3/20) * (12\pi)^{2/3} \approx 1.686 \tag{5.130}$$

5.9.2 Press Schechter theory

This is one of the most common way of getting the *abundance* of collapsed objects at any time from a given initial density field. This prescription is based on the following assumption:

> The fraction of mass in halos more massive than M is related to the fraction of volume in which the smoothed initial density field is above some threshold δ_c.

Apart from the above assumption it assumes that at very large scale linear perturbation is valid.

It works in the following sequence:

- Take any initial density field and smooth it over a *filter* of some CHARACTERSTIC size R.

- Find the fraction of volume in which density contrast δ is greater than some critical density δ_c.

  In general, if we consider a Gaussian density field the the probability of a region having density contracts in the range $[\delta, \delta + d\delta]$ will be as follows:

$$P(\delta, R, t_i) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma(R, t_i)}\exp\left[-\frac{\delta^2}{2\sigma^2(R, t_i)}\right] \tag{5.131}$$

  Where $\sigma^2(R, t_i)$ is defined as follows:

$$\sigma^2(R, t_i) = \int \frac{dk}{k}\left(\frac{k^3 P(k, t_i)}{2\pi^2}\right)\left[3\left(\frac{\sin kR - kR\cos kR}{k^3R^3}\right)\right]^2 \tag{5.132}$$

- From the above expression we can find the number density of objects greater than mass M by equating the fraction of volume for which $\delta > \delta_c$ to the the number of particles which are part of a collapsed object of mass $> M$. This gives us a function $F(M)$, which is nothing but the number density of objects having mass greater than M.

$$F(M) = \int_{\delta_c(t,t_i)}^{\infty} P(\delta, M, t_i) d\delta = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma(M)} \int_{\delta_c}^{\infty} \exp\left[-\frac{\delta^2}{2\sigma^2(M)}\right] d\delta \tag{5.133}$$

Or

$$F(M) = \frac{1}{2}\left[1 - \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma(M)} \int_0^{\delta_c} \exp\left[-\frac{\delta^2}{2\sigma^2(M)}\right] d\delta\right]$$
$$= \frac{1}{2}\left[1 - erfc\left(\frac{\delta_c}{\sqrt{2}\sigma}\right)\right] = \frac{1}{2} erfc\left(\frac{\delta_c}{\sqrt{2}\sigma(M)}\right)$$

There has been defined a function $f(M)$ which gives the number density of objects in per logregthemin bins in M, by the following way:

$$f(M) = \frac{\partial F(M)}{\partial M}$$

In order to compute $f(M)$ we need to differentiate $F(M)$ which is in the form in differentiation under integration and for it we use the following formula:

$$\frac{\partial}{\partial z}\int_{a(z)}^{b(z)} f(x,z)dx = \int_{a(z)}^{b(z)} \frac{\partial f}{\partial z} dx + f(b(z), z)\frac{\partial b(z)}{\partial z} - f(a(z), z)\frac{\partial a(z)}{\partial z} \tag{5.134}$$

Press-Schechter formalism under estimate the mass function by a factor of two so we need to arbitrary multiply the RHS of equation (5.133) by a factor of two. So now the $F(M)$ becomes as:

$$F(M) = 1 - \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma(M)} \int_0^{\delta_c} \exp\left[-\frac{\delta^2}{2\sigma^2(M)}\right] d\delta \tag{5.135}$$

This can be written as:

$$F(M) = 1 - \frac{1}{\sqrt{\pi}} \int_0^{\frac{\delta_c}{\sqrt{2}\sigma(M)}} e^{-x^2} dx$$

Now using the above formula we can easily find:

$$f(M) = \frac{\partial F(M)}{\partial M} = \frac{1}{\sqrt{2\pi}} \left(\frac{\delta_c}{\sigma(M)}\right) \left(\frac{1}{\sigma(M)} \frac{\partial \sigma(M)}{\partial M}\right) \exp\left(\frac{\delta_c^2}{2\sigma^2(M)}\right) \tag{5.136}$$

We can compute the number density of objects in mass range $[M, M+dM]$ by multiplying this function by $\frac{\rho}{M}$ which gives:

$$N(M) = \frac{\rho}{M} \frac{1}{\sqrt{2\pi}} \left(\frac{\delta_c}{\sigma(M)}\right) \left(\frac{1}{\sigma(M)} \frac{\partial \sigma(M)}{\partial M}\right) \exp\left(\frac{\delta_c^2}{2\sigma^2(M)}\right) \tag{5.137}$$

Power law model:

For a power law model:

$$P(k) \propto k^n \tag{5.138}$$

and

$$\sigma^2(M) \propto M^{\frac{-(n+3)}{3}} \tag{5.139}$$

5.10 Fractals and self similarity

Self similarity refers basically to a symmetry of system under dilation (in either space or time)

If any structure or system does not have any scale and it looks same on all scales then the structure is called fractal. There are two necessary things for a fractal system (1)inherent hierarchical organization, and (2) self-similarity, i.e., the copies within copies within...motif.

A line and a plane are self similar although line is one dimensional and a plane is two dimensional. This can be checked easily. Let us break a line into four self similar segments, each with same length, and magnify any one by a factor of four we find the original line. By same way if we decompose a square into four self similar squares and then double the size of any one square we get original square. For square we have four self similar squares but magnication factor is still 2. Same is true for cube also but this time we divide cube into eight parts and magnify each cube by a factor of two

On the basis of above discussion we can define the dimension of self similarity as following.

If we divide a figure into n self similar pieces and after magnifying any one of the piece by m we recover original figure then the fractal dimension d is defined as

$$d = \frac{log(n)}{log(m)}$$

log(copies) = d × log(amplifications)

Using this formula we can find the fractal dimension for line ,square and cube.

Object	Dimensions	Divide by (n)	Magnify by (m)	d
Line	1	2	2	1
Square	2	4	2	2
Cube	3	8	2	3

Fractal dimension need not be integer for example for the the triangle (figure**??**) it is $\frac{log3}{log2} \approx= 1.58$

In a scale invariant fractal model, the statistics varies as the power of distance

$$N(< r) = Ar^d$$

where A is constant and d is fractal dimension. For a uniform spatial distribution $N(< r) \propto volume$ so $d = 3$

Two point correlation function $\xi(x)$ for galaxy distribution is found to be small at scales $r > r_0$, where $r_0 < \frac{c}{H_0}$ is some characteristic length. Analysis of the galaxy distribution within fractal models show that at small scales

$$\xi(r) = \left(\frac{r_0}{r}\right)^{\gamma}$$

where $\gamma = 1.77 \pm 0.04$ and $r_0 = 5.4 \pm 1h^{-1}$ Mpc

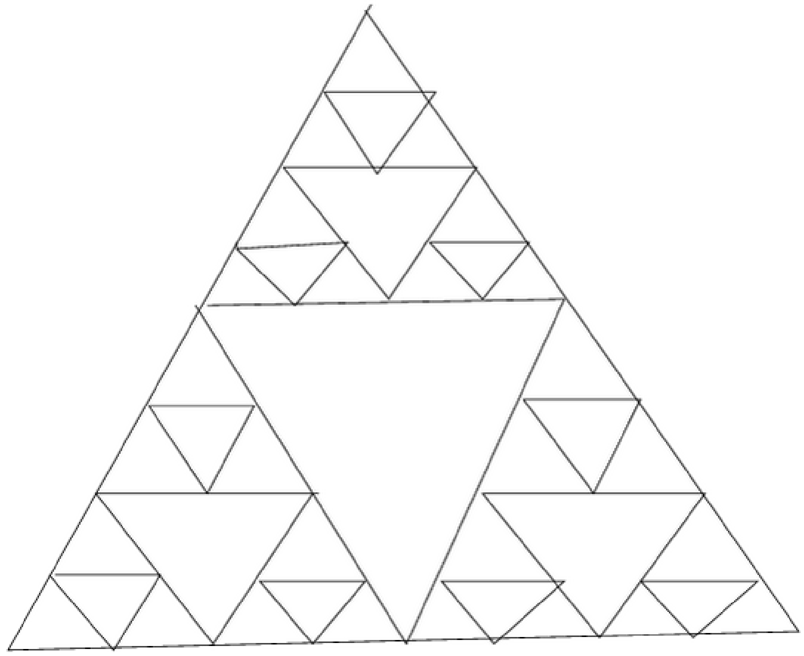


Figure 5.2: Sierpinski triangle consists 3 self similar pieces and after magnification by 2 we get original triangle so it's fractal dimension is $\frac{log(3)}{log(2)} = 1.58$

5.11 N-point correlation functions

Continiuos distribution of matter in an expanding universe is represented by discrete distribution

$$\rho(x,t) = ma^{-3}\sum_i \delta_D\left[x - x_i(t)\right] \tag{5.140}$$

Writing the Lagrangian for the system we can find the equation of motion and the force equation

$$\frac{dx^\alpha}{dt} = \frac{v^\alpha}{a} = \frac{p^\alpha}{a^2} \tag{5.141}$$

and the force equation

$$\frac{dp^\alpha}{dt} = -m\frac{\partial\phi}{\partial x^\alpha} \tag{5.142}$$

where the potential ϕ can be calculated by solving the Poisson equation in an expanding universe

$$\nabla^2\phi(x) = 4\pi G a^2\left[\rho(x) - \bar{\rho}\right] = 4\pi G a^2\bar{\rho}\delta(x) \tag{5.143}$$

In these equations a is the cosmic scale factor and the superscript α represents the labeling of statistical realization. It is easy to write the solution for the equation (5.143)

$$\phi(x) = Ga^2\bar{\rho}\int d^3x'^3\frac{\delta(x')}{x'} \tag{5.144}$$

and

$$-\frac{\partial \phi}{\partial x^{\alpha}} = Ga^2 \int [\rho(x) - \bar{\rho}] \frac{x'^{\alpha} - x^{\alpha}}{|x - x'|^3} d^3 x' \tag{5.145}$$

and for the discrete distribution

$$-\frac{\partial \phi}{\partial x^{\alpha}} = \frac{Gm}{a} \sum_j \frac{x_i^{\alpha} - x^{\alpha}}{|x_j - x|^3} \tag{5.146}$$

Consider a system of large number of interacting particles. In phase space position of any particle is given by its spatial position and momentum (x, p). We can define the phase density $f_1(x, p, t)$ (single particle distribution function) in terms of probability of finding a particle in the phase space volume $d^x d^3$ about the coordinates (x, p)

$$dP(x_1, p_1, t_1) = f_1(x_1, p_1, t) d^3 x_1 d^3 p_1 \tag{5.147}$$

Probability function P must satisfy the normalization condition

$$\int dP = 1 \tag{5.148}$$

one point distribution function can be directly related to the spatial density

$$\int f(x_1, p_1, t) d^3 p_1 = \rho(x_1, t) = na^3 \tag{5.149}$$

where $\rho(x_1)$ is the spatial density at point x_1 at time t and n is the number density of objects.

We can define two particle distribution function $f_2(x_1, p_1, x_2, p_2, t)$ in terms of joint probability of finding one particle in phase volume $d^3 x_1 d^3 p_1$ about (x_1, p_1) and another particle in phase volume $d^3 x_2 d^3 p_2$ about (x_2, p_2)

$$dP(x_1, p_1, x_2, p_2, t_1) = f_2(x_1, p_1, x_2, p_2, t) d^3 x_1 d^3 p_1 d_2^3 d^3 p_2 \tag{5.150}$$

Two point distribution function can be decomposed in terms of one point distribution functions

$$f_2(x_1, p_1, x_2, p_2, t) = f_1(x_1, p_1, t) f_2(x_2, p_2, t) + c(x_1, p_1, x_2, p_2, t) \tag{5.151}$$

or

$$f_2(1, 2) = f_1(1) f_1(2) + c(1, 2)$$

If the particles are uncorrelated (Poisson noise) then $c = 0$. For a collisionless system $c = 0$. A system of two particles can be treated as collisionless if the potential energy of their mutual interaction is much smaller then their kinetic energy. We can define two point correlation function ξ in terms of c

$$\int d^3 p_1 d^3 p_2 c(1, 2) = n^2 a^6 \xi(|x_2 - x_1|, t) \tag{5.152}$$

We can generalize this procedure for higher correlation functions also. For the three point correlation function

$$dP(x_1, p_1, x_2, p_2, x-3, p_3 t_1) = f_3(x_1, p_1, x_2, p_2, x_3, p_3 t) d^3 x_1 d^3 p_1 d_2^3 d^3 p_2 d^3 x_3 d^3 p_3 \tag{5.153}$$

Three point distribution function can be decomposed in terms of one point and two point distribution functions

$$f_3(1,2,3) = f_1(1)f_1(2)f_1(3) + f_1(1)f_2(2,3) + f_1(2)f_2(2,3)$$
$$+f_1(3)f_2(1,2) + d(1,2,3)c(x_1,p_1,x_2,p_2,t)$$

with the condition

$$\int d^3p_1 d^3p_2 d^3p_3 d(1,2,3) = n^3 a^9 \zeta(1,2,3) \tag{5.154}$$

here $\zeta(1,2,3)$ is the spatial three point correlation function.

BBGKY hierarchy:

Time evolution of any order of distribution function is governed by the Liouville's theorem. For the one point distribution function (particle density)

$$\left[\frac{\partial}{\partial t} + \dot{x}.\frac{\partial}{\partial x} + \dot{p}.\frac{\partial}{\partial p}\right] f_1(x,p,t) = 0 \tag{5.155}$$

Using the equation of motion and force equation for a particle at position (x_1, p_1, t) this can be written as

$$\left[\frac{\partial}{\partial t} + \frac{p_1^\alpha}{ma^2}\frac{\partial}{\partial x_1^\alpha} + \frac{dp_1^\alpha}{dt}\frac{\partial}{\partial p_1^\alpha}\right] f_1(x_1,p_1,t) = 0 \tag{5.156}$$

We can average this equation over configurations (ensemble)

$$< f_1(x_1,p_1,t) >= f_1(p_1,t) \tag{5.157}$$

$$< \frac{\partial f_1}{\partial x_1} >= \frac{\partial f_1(p_1,t)}{\partial x_1^\alpha} = 0 \tag{5.158}$$

Now from the dynamics we know

$$-\frac{\partial p_1}{\partial t} = \frac{\partial \phi}{\partial x_1}$$
$$= \frac{Gm}{a}\int f_1(x_2,p_2,t) d^3p_2 d^3x_2 \frac{(x_2 - x_1)}{|x_2 - x_1|^3}$$

This can be simplified [1] so average of the last term will be

$$< \frac{\partial p_1^\alpha}{dt}\frac{\partial f_1(x_1,p_1,t)}{\partial p_1^\alpha} >= \frac{Gm^2}{a}\frac{\partial}{\partial p_1} < \int f_1(x_2,p_2,t) d^3p_2 d^3x_2 \frac{(x_2 - x_1)}{|x_2 - x_1|^3} f_1(x_1,p_1,t) > \tag{5.159}$$

$$= \frac{Gm^2}{a}\frac{\partial}{\partial p_1}\int < f_1(x_2,p_2,t) f_1(x_1,p_1,t) > d^3p_2 d^3x_2 \frac{(x_2 - x_1)}{|x_2 - x_1|^3}$$

[1] For a general case we know the solution of the Possion's equation is

$$\phi(x_1) = \int \frac{\rho(x_2)}{x_2 - x_1} d^3x_2$$

$$\delta(x_2) = mn(x_2) = \frac{m}{a^3}\int f_1(x_2,p_2,t) d^3p_2$$

$$= \frac{Gm^2}{a}\frac{\partial}{\partial p_1}\int f_2(x_1, p_1, x_2, p_2, t) d^3p_2 d^3x_2 \frac{(x_2 - x_1)}{|x_2 - x_1|^3}$$

from the equations (5.156),(5.157),(5.158) and (5.159) we get

$$\frac{\partial f_1(p_1, t)}{\partial t} + \frac{Gm^2}{a}\frac{\partial}{\partial p_1^\alpha}\int f_2(1, 2) d^3p_2 d^3x_2 \frac{x_{21}}{|x_{21}|^3} = 0$$

here $x_{21} = x_2 - x_1$ This equation further can be simplified

$$\frac{\partial f_1(p_1, t)}{\partial t} + \frac{Gm^2}{a}\frac{\partial}{\partial p_1^\alpha}\int [f_1(1)f_1(2) + c(1, 2)] d^3p_2 d^3x_2 \frac{x_{21}}{|x_{21}|^3} = 0$$

since we know that

$$\int d^3x^2 \frac{x_{21}}{|x_{21}|^3} = 0$$

so the equation simplified as

$$\frac{\partial f_1(p_1, t)}{\partial t} + \frac{Gm^2}{a}\frac{\partial}{\partial p_1^\alpha}\int c(1, 2) d^3p_2 d^3x_2 \frac{x_{21}}{|x_{21}|^3} = 0 \tag{5.160}$$

This is the first **BBGKY** equation, it shows that the evolution of the one point distribution function (density) is determined by the two point correlation functions (two point distribution function).

This procedure can be generalized for the higher order distribution also for the two point distribution function

$$\left[\frac{\partial}{\partial t} + \frac{p_1^\alpha}{ma^2}\frac{\partial}{\partial x_1^\alpha} + \frac{dp_1^\alpha}{dt}\frac{\partial}{\partial p_1^\alpha} + \frac{\partial}{\partial t} + \frac{p_2^\alpha}{ma^2}\frac{\partial}{\partial x_2^\alpha} + \frac{dp_2^\alpha}{dt}\frac{\partial}{\partial p_2^\alpha}\right] f_2(x_1, p_1, x_2, p_2, t) = 0 \tag{5.161}$$

Again using the fact that

$$\frac{\partial p_1^\alpha}{\partial t} == \frac{Gm}{a}\int f_1(x_3, p_3, t) d^3p_3 d^3x_3 \frac{(x_3 - x_1)}{|x_3 - x_1|^3} \tag{5.162}$$

and

$$< f_2(1, 2) f_1(3) >= f_3(1, 2, 3)$$

On averaging the equation (5.161) we get

$$\frac{\partial f_2(1, 2)}{\partial t} + \frac{p_1^\alpha}{ma^2}\frac{\partial f_2(1, 2)}{\partial x_1^\alpha} + \frac{Gm^2}{a}\frac{x_{21}^\alpha}{x_{21}^3}\frac{\partial f_2(1, 2)}{\partial p_1^\alpha} +$$

$$\frac{Gm^2}{a}\frac{\partial}{\partial p_1^\alpha}\int d^3x_3 d^3p^3 \frac{x_{31}^\alpha}{{x_{31}}^3} f_3(1, 2, 3) + 1 \longleftrightarrow 2 = 0 \tag{5.163}$$

This equation shows that the evolution of f_2 depends on f_3. This hierarchy exists for all order and is called the **BBGKY hierarchy**.

The general form of BBGKY equation is as follows

$$\frac{\partial f_n}{\partial t} + \frac{p_i}{ma^2}\frac{\partial f_n}{\partial x_i} + \int d^3x_{n+1}d^3p_{n+1}\frac{Gm^2}{a}\frac{\partial f_{n+1}}{\partial p_i}\frac{\partial}{\partial x_i}\frac{1}{|x_{n+1}-x_i|} = 0 \tag{5.164}$$

The relation between the n-point distribution function and n-point reduced correlation functions is as following

$$\int d^3p_1......d^3p_n f_n = (na^3)^n \zeta_n(x_1, x_2, ..x_n) \tag{5.165}$$

where $\zeta_n(x_1, x_2, ..x_n)$ are reduced n-point correlation functions.

$\zeta_n(x_1, x_2, ..x_n)$ as a connected parts of point density correlations:

We can define

$$\zeta_n(x_1, x_2, ..x_n) =< \delta(x_1)\delta(x_2)......\delta(x_n) >_c =< \delta(x_1)\delta(x_2)......\delta(x_n) > -\sum_{S\epsilon P(x_1,..x_n)}\Pi\zeta \tag{5.166}$$

Meaning of the connected parts is clear from the following diagram

$$< \delta_1\delta_2\delta_3\delta_4 >=$$

Pair conservation equation & self similarity:

With the help of equation (5.163) we can write an equation for the conservation of pairs

$$\frac{\partial \xi}{\partial t} + \frac{1}{x^2 a}\frac{\partial}{\partial x}\left[x^2(1+\xi)v\right] = 0 \tag{5.167}$$

Similarity solution is based on the time scaling in the Einstein-de Sitter universe

$$H = \frac{2}{3t}; 8\pi G\rho = \frac{4}{3t^2}$$

Scaling relation for ξ is

$$\xi(x,t) = \xi\left(\frac{x}{t^\alpha}\right) \tag{5.168}$$

This equation shows that ξ is the function of scales variable $\frac{x}{t^\alpha}$

It is easy to show that in linear limit

$$\xi(x,t) \propto a^2\xi(x,0) \propto t^{\frac{4}{3}}x^{-(n+3)} = \xi(\frac{x}{t^\alpha}) = \left(\frac{x}{t^{\frac{4}{(9+3n)}}}\right)^{-(n+3)} \tag{5.169}$$

so $\alpha = \frac{4}{(9+3n)}$. Here n is the index of power spectrum.

We can obtain similarity solution in the non-linear case with the help of pair conservation equation is we assume that at small x, where $\xi >> 1$, the particle pairs are in the same gravitationally bound and stable cluster. This implies that the mean rate of proper separation of the pairs is close to zero or

$$v = -\frac{2a}{3t}x; \xi >> 1 \tag{5.170}$$

Putting this in pair conservation equation we get

$$\frac{\partial \xi}{\partial t} = \left(\frac{2\xi}{t}\right) \tag{5.171}$$

For any scale when $\delta \sim 1$ then that scale become nonlinear

$$\delta \propto t^{\frac{2}{3}} x^{-\frac{(n+3)}{2}}$$

so the scale at which $\delta = 1$ at time t is

$$x_0(t) = t^{\frac{4}{9+3n}} \tag{5.172}$$

This relation can be written in terms of physical distance $r = ax$ also

$$r_0(t) = a(t)x_0(t) = t^{\frac{2n+10}{9+3n}} \tag{5.173}$$

since we know that

$$s = \frac{x}{t^{\alpha}} = \frac{ax}{t^{\alpha+\frac{2}{3}}} = \frac{r}{t^{\alpha+\frac{2}{3}}}$$

which gives

$$\frac{\partial s}{\partial t} = -\frac{(\alpha + \frac{2}{3})}{t}$$

so

$$\frac{d\xi}{ds} = \frac{\partial \xi}{\partial t}\left(\frac{\partial t}{\partial s}\right) = -\left(\frac{2}{\alpha + \frac{2}{3}}\right)\frac{\xi}{s}$$

or

$$\xi \propto s^{-\gamma}; \gamma = \left(\frac{3n+n}{5n+3}\right) \tag{5.174}$$

for $n = 0$ we get $\xi \propto s^{-1.8}$

5.12 HKLM method

The average number of neighbors of a particle enclosed within a constant Lagrangian distance r_0 remains constant. This can be given in the form of the pair conservation equation (Peebles 1976; Peebles 1977).

$$\frac{\partial \xi(x,t)}{\partial t} + \frac{1}{ax^2}\frac{\partial}{\partial x}\left(x^2[1+\xi(x,t)]v\right) = 0 \tag{5.175}$$

The above equation shows that the rate of change of the mean number of neighbors within a distance x of a particle is equal to the mean flux of neighbors crossing the constant x surface (Peebles 1980).

$$\frac{\partial}{\partial t}\left(\bar{n}a^3 \int_0^r 4\pi r^2 dr[1+\xi(r,t)]\right) + 4\pi a^2 r^2 \bar{n}[1+\xi(r,t)]v = 0 \tag{5.176}$$

Here $\bar{n}$ is the average number density of particles and v is the radial velocity i.e., velocity along the direction of the line joining particles, of a pair of particles which has two components one due to hubble expansion and another due to peculiar motions.

If Lagrangian (initial) coordinate of a conserved pair is r_0 then from equation (5.176)

$$r_0^3 = \int_0^r d^3r'[1+\xi(r',t)] = r^3[1+\bar{\xi}(r,t)] \tag{5.177}$$

where $\bar{\xi}$ is the average two point correlation function which is defined as follows

$$\bar{\xi}(r) = \frac{1}{r^3}\int_0^r d^3r'\xi(r') \tag{5.178}$$

Note that:

- In linear limit:
$$\bar{\xi} \propto a^2 \tag{5.179}$$
- In highly nonlinear limit when clustering becomes stationary (stable) then for a fixed r_0
$$\bar{\xi} \propto a^3 \tag{5.180}$$
This is because for a constant r_0, r decreases as $1/a^3$.

HKLM (Hamilton et. al. 1991) hypothesis:

The time evolution of the physical radius r of a conserved pair surface $r_0 = constant$ in a $\Omega = 1$ universe is a universal function of cosmic scale factor.

$$\bar{\xi} = \bar{\xi}\left(a^2\bar{\xi}_0(r_0)\right) \tag{5.181}$$

where $\bar{\xi}_0$ is determined by initial conditions, $\bar{\xi} = a^2\bar{\xi}_0(r_0)$ i.e., linear limit. Note that for a power law model:

$$\bar{\xi}_0(r_0) \propto r_0^{-(n+3)} \tag{5.182}$$

The above relationship can be checked in N-body simulations.

From equation (5.177) we can find a relationship between the scaled pair velocity i.e., v/Hr and average two point correlation function.

$$\frac{v}{-Hr} = -\left.\frac{dlnr}{dlna}\right|_{r_0} = \left.\frac{dln(1+\bar{\xi})}{dlna}\right|_{r_0} \tag{5.183}$$

Note that since in linear limit $\bar{\xi} \propto a^2$ so $-v/Hr = 2/3$.

HKLM and many groups onwards have proposed fits for HKLM relation (equation (5.181)) which correctly reproduce the asymptotic limits.

5.13 Lagrangian and Eulerian approach of dynamical systems

There are two important approaches, named Lagrangian and Eulerian, to study the motion of a fluid. In the first approach, we follow the motion of individual fluid particles which are labeled by some labels, let us say by q. In this approach, once we know the trajectory of a particle we can compute its velocity, density in its neighborhood etc. In the second approach, we measure the physical properties of particles at various space-time points.

Let us consider a scaler quantity $S(x, y, z, t)$ for which space and time derivatives exist. In general, the total rate of change of S with time is given by

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x}\frac{dx}{dt} + \frac{\partial S}{\partial y}\frac{dy}{dt} + \frac{\partial S}{\partial z}\frac{dz}{dt}$$

or

$$\frac{DS}{Dt} = \frac{\partial S}{\partial t} + \vec{v}.\nabla S \tag{5.184}$$

- The term $\frac{\partial S}{\partial t}$ represents the local time rate of change of the quantity S at a fixed point (x, y, z).
- Here $\vec{v}.\nabla S$ is a scalar and represents the advection or change in the field with the motion of fluid. This term is zero only when either $\vec{v} = 0$ or∇S or both are perpendicular to each other.

For the physical picture of above description let us take consider a point$A(x_A, y_A, z_A, t)$ at which $S_A = S(x_A, y_A, z_A, t)$. Now $\left(\frac{DS}{Dt}\right)_A$ represents the time rate of change of S in the fluid in the small parcel A.

If we are interested in the time rate of change of a physical quantity for a particular parcel of fluid, it is somewhat easier to compute $\left(\frac{DS}{Dt}\right)$ and then evaluate it at A rather than evaluate $S_A(t)$ and then its time derivative. The former approach is the Eulerian and the later is Lagrangian.

Let us summarize these two method in the following way.

- **Eulerian Method** :- We must specify velocity field and then take convective derivative $\left(\frac{DS}{Dt}\right)$ of the field S
- **Lagrangian Method** :- Particles trajectories must be specified.

Example :- Given that velocity field $u(x, t) = -ax + bt$ find the total rate of change of the scaler field $S(x, t) = S_0 sin(\omega t) + mt$ at a point A **solution** :- 1. Lagrangian method ;- From the $u(x, t) = -ax + bt$ web get the trajectory of particle A one boundry conditions are given. Let they are $\dot{y} = \dot{z} = 0$. we get

$$x_A = Ce^{-at} + \frac{b}{a^2}(at - 1) \tag{5.185}$$

with

$$C = \frac{b}{a^2} + x_A(0) \tag{5.186}$$

now putting the value of x_A we get field at A as following

$$S_A = S_0 sin(\omega t) = mce^{-at} + \frac{mb}{a^2}(at - 1) \tag{5.187}$$

and

$$\frac{dS_A}{dt} = wS_0 cos(\omega t) + mCe^{-at} + m\frac{b}{a^2}(at - 1) = wS_0 cos(\omega t) + mu_A \tag{5.188}$$

2. Eulerian method : We directly compute $\frac{Dt}{Dt}$ using velocity field and field

$$\frac{Dt}{Dt}_A = wS_0 cos(\omega t) + mu_A$$

Mass conservation:

It is easy to follow various conservations laws in Lagrangian approach than in Eulerian approach. However, we can always transform various physical quantities from Lagrangian coordinates to Eulerian coordinates and vice versa.

Let us consider a fluid particle which has its end points at α_1 and α_2 at initial time t_0 and ξ_1 and ξ_2 at some final time t. Now if its cross section is A and density is $\rho_0(\alpha)$ at time t_0 and $\rho(\alpha, t)$ at time t then its mass:

$$M = A\int_{\alpha_1}^{\alpha 2} \rho_0(\alpha) d\alpha A \int_{\xi_1}^{\xi_2} \rho(\alpha, t)$$

Now from the Mean Value Theorem:

$$M = A\rho_0(\alpha)(\alpha_2 - \alpha_1) = A\rho(\alpha, t)(\xi_2 - \xi_1)$$

or

$$\rho(\alpha, t) = \frac{(\alpha_2 - \alpha_1)}{(\xi_2 - \xi_1)}\rho_0(\alpha) \approx \left(\frac{\partial \xi}{\partial \alpha}\right)^{-1} \rho_0(\alpha)$$

Lagrangian position of a fluid element does not change with time; the Eulerian coordinates at some fixed time t_0 would be a set of Lagrangian coordinates. In a homogeneous and isotropic universe (unperturbed universe) Lagrangian coordinates are same as the comoving coordinates.

5.14 Approximation Methods

5.14.1 Zeldovich Approximation

Zeldovich approximation is a first order Lagrangian perturbation the theory. In a homogeneous and isotropic universe Lagrangian coordinate q (every particle is labeled by some q) and Eulerian coordinate r are related by the following way:

$$r(t) = a(t)q$$

Now Zeldovich approximation begins with considering that in linear regime this relation can be modified in the following way.

$$r(t) = a(t)x(t) = a(t)q + b(t)p(q) = a(t)[q + D_+(t)p(q)] \tag{5.189}$$

Here $D_+(t)$ is the growing mode of perturbation and $p(q)$ is related to initial perturbation.

Before proceeding further firstly we must prove that the equation (5.189) matches with the results of linear perturbation theory (LPT). In order to do that let compute the density from equation (5.189) using mass conservation:

$$\rho_0 d^3q = \rho(x,t)d^3x$$

$$\rho(x,t) = \frac{\rho_0}{J\left[\frac{\partial x}{\partial q}\right]} = \frac{\rho_0/a^3}{\left|\delta_{ij} + D_+(t)\frac{\partial p_j}{\partial q_j}\right|}$$
$$= \rho_b(t)[1 - D_+(t)\nabla p(q)]$$

Let us consider the following tensor (called the deformation tensor):

$$D_{ij} = \frac{\partial x_i}{\partial q_j} = \delta_{ij} + D_+(t)\partial_j p_i(q)$$

Now we can chose our coordinate axis by such a way that this matrix transforms into a diagonal matrix.

$$D_{ij} = \begin{bmatrix} 1 - D_{+1}(t)\lambda_1(q) & 0 & 0 \\ 0 & 1 - D_+(t)\lambda_2(q) & 0 \\ 0 & 0 & 1 - D_+(t)\lambda_3(q) \end{bmatrix}$$

so now we can write:

$$\rho(r,t) = \frac{\rho_b(t)}{[1 - D_+(t)\lambda_1(q)][1 - D_+(t)\lambda_2(q)][1 - D_+(t)\lambda_3(q)]}$$

If all λ_1, λ_2 and λ_3 are positive then the collapse occurs in the form of a singularity, howere, all three are negative then we get a void. In the case when any two λs are positive and one is negative then there forms a filaments. Note that the collapse occurs firstly along the axis for which λ is maximum. In general it is very unlike that all λ have the same value so if $\lambda_1 > \lambda_2 > \lambda_3$ then firstly collapse occurs along the axis and two dimensional structure forms which is called a *pancake*.

Now we can write:

$$\rho(r,t) \approx \rho_b(t)\left[1 + D_+(t)(\lambda_1 + \lambda_2 + \lambda_3)\right]$$

Note that in linear perturbation theory we have:

$$\rho(x,t) = \rho_b(t)(1 + \delta(x,t)) = \rho_b(t)(1 + D_+(t)\delta(x,0))$$

Now we can compare the LPT and Zeldovich approximation:
From LPT:

$$\nabla_x^2\phi = 4\pi G\rho_b a^2\delta \tag{5.190}$$

and from ZA we have:

$$\rho(x,t) = \rho_b(t)[1+\delta(x,t)] = \rho_b(t)[1 - D_+(t)\nabla_q p(q)]$$

so this means that $\delta(x,t) = -D_+(t)\nabla_q p(q)$. Now if

$$p(q) = \nabla_q \phi_0(q)$$

then using equation (5.190)

$$\phi_0(q) = -\frac{\phi}{4\pi G\rho_b D_+ a^2}$$

Now using $4\pi G\rho_b = -3\ddot{a}/a$ and $D_+(t) = b(t)/a(t)$. so

$$\phi_0(q) = -\frac{\phi}{3\ddot{a}ab} \quad \text{or} \quad \phi = -3\ddot{a}ab\phi_0 \tag{5.191}$$

In order to see the physical significance of this approximation let us write:

$$x(t) = q + D_+(t)p(q)$$

Let us use $D_+(t) = \tau$ as time parameter and identify $p(q)$ with the peculiar velocity $u(q)$ of the particle labeled by q and write $q = x(t_0) = x_0$:

$$x(\tau) = x_0 + u\tau$$

This equation describes the motion along a straight line. This means that in the Zeldovich Approximation particles moves along straight alone with a constant velocity $u = \nabla\phi_0$.

we know that the equation of particle can be written in the following way also:

$$\frac{dw}{db} = -\frac{3}{2b}A(w + \nabla\psi)$$

Here $w = dx/db$ is the peculiar velocity. From the above equation it is clear that in Zeldovich approximation RHS i.e., is zero:

$$\frac{dw}{db} = -\frac{3}{2b}A(w + \nabla\psi) = 0$$

or

$$w = -\nabla\psi$$

note that:

$$\psi = \frac{2}{3H_0^2}\left(\frac{a}{b}\right)\phi$$

and

$$\phi_0(q) = -\frac{\phi}{4\pi G\rho_b D_+ a^2} = -\frac{1}{4\pi G\rho_b a^2 \frac{b}{a}}\phi = \frac{2}{3H_0^2}\left(\frac{a}{b}\right)\phi = -\psi$$

Note that Zeldovich approximation remains valid as long as the acceleration term is zero i.e., $dw/db = 0$. This approximation is exact in one dimensions as long trajectories of particles do not cross each other since in this case gravitational force is independent from the distance between particles. The gravitational force acting on a particle in one dimension just depends on how many particles are on its either side.

5.14.2 Adhesion Approximation

The main drawback of Zeldovich approximation is that it fails to bound gravitating particles when they come very close to each other. This is because in ZA the trajectories of particles are straight lines, however, in reality we expect particles to oscillate around the minima of potential, in place of going along the straight line and making the pancake thick. In order to account for the confinement particles there have been proposed many modification and one of them is called the *Adhesion approximation*(AA).

Adhesion approximation directly follows from the Zeldovich approximation, however, we include an artificial velocity in order to take into account the gravity. So in this case when particles approach each other they stick together in place on going along the straight line.

Equation of motion in comoving coordinates:

$$\frac{du}{dt} + \frac{\dot{a}}{a}u = -\frac{\nabla_x \phi}{a}$$

Where $u = v - Hr = \dot{r} - \dot{a}x$ and $r(t) = a(t)x(t)$.
and continuity equation:

$$\frac{\partial \delta}{\partial t} + 3\frac{\dot{a}}{a}\rho + \frac{1}{a}\nabla(\rho u) = 0$$

Possion's equation:

$$\nabla^2 \phi = 4\pi G \rho_b a^2 \delta$$

Now from Zeldovich approximation:

$$r(t) = a(t)x(t) = a(t)[q + b(t)p(q)]$$

Where

$$a\ddot{b} + 2\dot{a}\dot{b} + 3\ddot{a}b = 0$$

and

$$p(q) = \nabla \phi_0(q) \quad \text{and} \quad \phi_0 = \phi/3\ddot{a}ab$$

Now from Zeldovich approximation:

$$\frac{du}{dt} = \frac{d}{dt}(a\dot{x}) = \frac{d}{dt}[a\dot{b}p(q)] = \left(\frac{\dot{a}}{a} + \frac{\ddot{b}}{\dot{b}}\right) u$$

Now using two new variables $\tilde{u} = u/\dot{a}a$; and $\eta = \eta = a^3\rho$ and using the scale factor a as time variable we can write the above equation and the continuity equation in the following form:

$$\frac{d\tilde{u}}{da} = \frac{\partial\tilde{u}}{\partial a} + \tilde{u}(\nabla\tilde{u}) = -\frac{1}{\dot{a}}\left(\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b}\right)\tilde{u} \tag{5.192}$$

and

$$\frac{\partial\eta}{\partial a} + \nabla(\eta\tilde{u}) = 0 \tag{5.193}$$

Note that for Einstein-de Sitter model $b = a$ so RHS of equation (5.192) which appears like a viscus term is zero.

In Adhesion approximation we modify the above equation of motion in the following way.

$$\frac{\partial\tilde{u}}{\partial b} + \tilde{v}_k\frac{\partial\tilde{u}_i}{\partial x_k} = \nu\frac{\partial^2\tilde{u}_i}{\partial x_k^2}$$

Considering that the velocity $\tilde{u}$ is irrotational i.e., $\tilde{u} = \nabla\Phi$ we can write the above equation in the following form:

$$\frac{\partial\Phi}{\partial b} + \frac{1}{2}\left(\frac{\partial\phi}{\partial x_i}\right) = \nu\frac{\partial^2\tilde{u}_i}{\partial x_k^2}$$

Let us make the substitution:

$$\Phi = -2\eta lnU$$

then

$$\frac{\partial U}{\partial b} = \nu\frac{\partial^2 U}{\partial x_i^2}$$

and we get the final solution:

$$\tilde{u}(x,b) = \frac{\int \frac{x-q}{b}\exp\left[-\frac{1}{\nu}G(x,q,b)\right]d^3q}{\int\exp\left[-\frac{1}{\nu}G(x,q,b)\right]d^3q} \tag{5.194}$$

Where

$$G(x,q,b) = \phi_0(q) + \frac{(x-q)^2}{2b}$$

Note that $\tilde{}(x,0) = p(q) = \nabla\phi_0$ and when $\nu =$ we get

$$\tilde{u} = \frac{x - q(x,b)}{b}$$

which is nothing but Zeldovich approximation.

Chapter 6

N-Body simulations

N-body simulations are the most important tools to understand non-linear gravitational clustering in an expanding universe. Non-baryonic dark matter is discreatized in the form of a finite number of particles and their motion under gravity and expansion of the background is followed by integrating their equation of motion. In cosmological N-body simulations a single particle reprsents a large number of actual dark matter particles so numerical artifacts like two body collisions are important which should be handled carefully. Cosmological N-body simulations are finite in volume as opposite to the actual universe which is infinite so periodic boundary conditions are commonly used. However, care is taken to ensure that the distribution of the simulation particles is always smooth at the largest scales in accordance to the actual universe which is on an average smooth on scales larger than 100 Mpc.

6.1 Equation of motion

The equations of motion of a system of N gravitating particles in comoving coordinates is given by

$$\ddot{x}_i + 2\frac{\dot{a}}{a}\dot{x}_i = -\frac{\nabla_x \phi}{a^2} \tag{6.1}$$

or

$$\frac{dv_i}{dt} + Hv_i = g \tag{6.2}$$

where

$$v_i = a\frac{dx_i}{dt} \quad \text{and} \quad g = -\frac{\nabla_x \phi}{a} \tag{6.3}$$

From equation (6.2) we can observe that the expansion of the universe, which is represented by cosmic scale factor $a(t)$ or the Hubble parameter $H = \dot{a}/a$, acts as a drag force on particles. The velocity dependent term can be absorbed, if we write the equation of motion in terms of a new variable s which is defined as follows

$$s = H_0 \int \frac{dt}{a^2} \tag{6.4}$$

In this case the equation of motion is

$$\frac{d^2x}{ds^2} = a^3 g \tag{6.5}$$

Gravitational potential ϕ which is contributed by the density fluctuations $\delta(x,t) = [\rho(x,t) - \rho_b(t)]/\rho_b(t)$, where $\rho_b(t)$ is the background density, is given by

$$\nabla_x^2 \phi(x,t) = 4\pi G \rho_b(t) a^2 \delta(x,t) = \frac{3H_0^2}{2} \Omega_{NR}(t = t_0) \frac{\delta}{a} \tag{6.6}$$

or

$$\nabla^2 \psi = \frac{\delta}{a} \quad \text{where} \quad \psi = \frac{2}{3H_0^2 \Omega_{NR}} \phi \tag{6.7}$$

Now in order to evolve the trajectories of the particles we need to solve the following system of equations consistently.

$$\frac{d^2x}{ds^2} = a^3 g \tag{6.8}$$

$$g = -\frac{\nabla_x \phi}{a} = -\frac{3H_0^2 \Omega_{NR}}{2a} \nabla \psi \tag{6.9}$$

$$\nabla^2 \psi = \frac{\delta}{a} \tag{6.10}$$

and

$$s = H_0 \int \frac{dt}{a^2} = H_0 \int \frac{da/a^3}{H(a)} \tag{6.11}$$

where for Einstein-de Sitter model

$$H(a) = H_0 \left[\Omega_{NR} \frac{1}{a^3} + \Omega_\Lambda + (1 - \Omega_0) \frac{1}{a^2} \right]^{1/2} \tag{6.12}$$

or

$$H(a) = \frac{H_0}{a} \left[1 + \left(1 - \frac{1}{a} \right) \Omega_{NR} + (a^2 - 1) \Omega_\Lambda \right]^{1/2} \tag{6.13}$$

6.1.1 Linear growth factor as time parameter

Sometimes it is more useful to use the growing solution $b(t)$ of the density contrast $\delta(x,t)$ as "time" variable.

In an expanding universe the evolution of density contrast is governed by the following equation

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G \rho_b \delta \tag{6.14}$$

and its growing solution is

$$b(t) \propto \frac{X^{1/2}}{a} \int_a \frac{da}{X^{3/2}} \quad \text{where} \quad X^{1/2} = aH/H_0 \tag{6.15}$$

In this case we need to solve the following set of equations

$$\frac{dx}{db} = u \tag{6.16}$$

$$\frac{du}{db} = -\frac{3Q}{b}(u - g) \tag{6.17}$$

$$g = -\nabla\psi \tag{6.18}$$

where

$$Q = \left(\frac{\rho_b}{\rho_c}\right)\left(\frac{\dot{a}b}{a\dot{b}}\right)^2 \tag{6.19}$$

and ψ is given by equation (6.7).

6.1.2 Dimensional variables

It is more useful to write the equation of motion in the following dimensionless variables.

$$\ddot{x} + 2\dot{a}\dot{x} = -\frac{\nabla_x \phi}{a} \tag{6.20}$$

Now if we define

$$p = a^2 x \quad \text{or} \quad v = p/a = a\dot{x} \tag{6.21}$$

then

$$\frac{dx}{da} = \frac{p}{\dot{a}a^2} \tag{6.22}$$

$$\frac{dp}{da} = -\frac{\nabla\phi}{\dot{a}} \tag{6.23}$$

and

$$\nabla^2\phi = \frac{3H_0^2\Omega_{NR}}{2}\frac{\delta}{a} \tag{6.24}$$

Now define the dimensional variables $\tilde{x}, \tilde{t}, \tilde{\phi}$ and $\tilde{\phi}$ in the following way

$$x = \tilde{x}x_0; t = (1/H_0)\tilde{t}, \phi = (x_0 H_0)^2\tilde{\phi} \quad \text{and} \quad \rho = (\rho_c \Omega_{NR}/a^3)\tilde{\rho} \tag{6.25}$$

then equation (6.22), (6.23) and equation (6.24) can be written as

$$\frac{d\tilde{x}}{da} = F(a)\frac{\tilde{p}}{a^2} \tag{6.26}$$

$$\frac{d\tilde{p}}{da} = -F(a)\tilde{\nabla}\tilde{\phi} \tag{6.27}$$

and

$$\tilde{\nabla}^2\tilde{\phi} = \frac{3}{2}\frac{\delta}{a} \tag{6.28}$$

where $\tilde{\nabla} = x_0\nabla$ and

$$F(a) = \frac{H_0}{\dot{a}} = \frac{H_0/a}{H(a)} \tag{6.29}$$

where $H(a)$ is given by equation (6.13).

6.2 Integrating equation of motion

We can update the positions and velocities of particles using Taylor series. If position, velocity and acceleration of a particle at time $t_n = t_0 + n\Delta t$ and $t_{n+1} = t_n + \Delta t$ are x_n, v_n, f_n and x_{n+1}, v_{n+1} and f_{n+1} respectively then

$$x_{n+1} = x_n + v_n(\Delta t) + \frac{f_n}{2}(\Delta t)^2 + O[(\Delta t)^3] \tag{6.30}$$

and

$$v_{n+1} = v_n + f_n(\Delta t) + \frac{j_n}{2}(\Delta t)^2 + O[(\Delta t)^3] \tag{6.31}$$

since we do not the rate of variation of acceleration (jerk) j_n so we can write it in terms of acceleration

$$j_n = \frac{f_{n+1} - f_n}{(\Delta t)} + O[(\Delta t)] \tag{6.32}$$

so

$$v_{n+1} = v_n + \frac{(\Delta t)}{2}(f_{n+1} + f_n) + O[(\Delta t)^3] \tag{6.33}$$

Since the number of integration steps are proportional to $1/\Delta t$ so this method is accurate up to $O[(\Delta t)^3] \times 1/(\Delta t) = O[(\Delta t)^2]$.

Leap-Frog methods:

In place of the above method Leap-Frog methods are more commonly used in the case when force is independent from velocity. In these methods velocities are computed at the mid points of integration time steps.

In this case

$$x_{n+1} = x_n + v_{n+1/2}\Delta t \tag{6.34}$$

and

$$v_{n+3/2} = v_{n+1/2} + f_{n+1}\Delta t \tag{6.35}$$

It can be easily checked that these methods are also $O[(\Delta t)^2]$ accurate.

$$x_{n+1} = x_n + v_n \Delta t + \frac{f_n}{2}(\Delta t)^2 + O[(\Delta t)^3]$$

substituting

$$f_n = \frac{v_{n+1/2} - v_n}{\Delta t/2} + O[(\Delta t)] \tag{6.36}$$

which gives

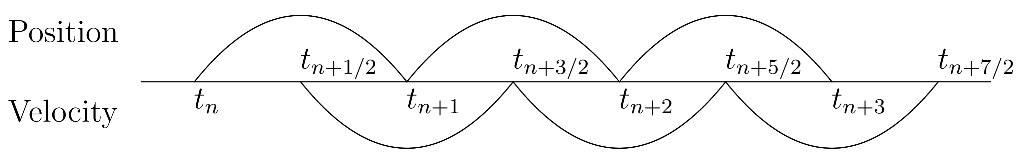
$$x_{n+1} = x_n + v_{n+1/2}\Delta t + O[(\Delta t)^3]$$

In the same way if we substitute

$$j_n = \frac{f_{n+1/2} - f_n}{\Delta t/2} + O[\Delta t]$$

then we find

$$v_{n+3/2} = x_n + f_{n+1}\Delta t + O[(\Delta t)^3]$$



6.2.1 Flow chart for a cosmological N-body simulation code