Dark Energy

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Abstract. The cosmological constant problem as well as the case for dark energy are briefly reviewed and some theoretical models of dark energy are discussed in detail. These include: the cosmological constant, quintessence, the Chaplygin gas and Braneworld models. I also discuss model independent measures of dark energy and conclude by mentioning some properties of the Statefinder diagnostic which can successfully differentiate between different families of dark energy models.

1 Introduction: The cosmological constant and vacuum energy

One of the most remarkable discoveries of the past decade is that the universe is accelerating.[1] Evidence for an accelerating universe comes from observations of high redshift type Ia supernovae treated as standardized candles [4] and, more indirectly, by observations of the cosmic microwave background and galaxy clustering [5, 6]. Perhaps the simplest explanation for acceleration (and indeed the oldest) is the presence of vacuum energy exhibiting itself as a small cosmological constant with equation of state $p = -\rho =$ constant. So let me begin this lecture series by first saying a few words about about the cosmological constant or Λ.

In 1917, only a few years after introducing the field equations of the General theory of relativity (GR), Einstein proposed adding a 'cosmological constant' to these equations which were modified to

$$R_{ik} - \frac{1}{2}\, g_{ik} R - \Lambda g_{ik} = \frac{8\pi G}{c^4} T_{ik}. \tag{1}$$

The main motivation behind introducing the cosmological constant appears to have been Einstein's belief that the equations of General Relativity should be compatible with Mach's principle. Einstein was fascinated by the arguments of philosopher/scientist Ernst Mach. Mach was concerned about the notion of absolute motion which prevailed in Newtonian mechanics. He postulated that all the matter in the universe including the distant stars provided a 'background' against which motion could be measured and that unless there was a material background which served as a reference frame, it was meaningless to talk of rest or motion in any absolute sense. Einstein proposed incorporating Mach's principle into the general theory of relativity by suggesting a solution of the equations (1) in which the universe was static and closed on itself, much like the closed two dimensional surface of a balloon. Einstein later discarded Λ after becoming convinced that the Universe was expanding, yet the cosmological constant has continued to hold a lasting fascination in the minds of physicists and cosmologists.

Although Einstein originally introduced the cosmological constant (Λ) into the left hand side of his field equations, it has now become conventional to show the Λ-term in the RHS, treating it as an effective form of matter. This indeed has been the rationale behind most dark energy models (though not all – see for instance the discussion of braneworld dark energy models in this section). In a homogeneous and isotropic Friedmann-Robertson-Walker (FRW) universe consisting of pressureless dust (dark matter) in addition to Λ the Raychaudhury equation, which follows from (1), takes the form

$$\ddot{a} = -\frac{4\pi G}{3} a\rho_m + \frac{\Lambda}{3}\ , \tag{2}$$

[1] This lecture series is largely based on the reviews [1, 2, 3] and the interested reader is referred to them for more details.

Equation (2) can be rewritten in the form of a force law:

$$\mathcal{F} = -\frac{GM}{R^2} + \frac{\Lambda}{3}R, \quad (R \equiv a) \tag{3}$$

which demonstrates that the cosmological constant gives rise to a *repulsive* force whose value increases with distance, and which could therefore be responsible for the current acceleration of the universe.

Although introduced into physics in 1917, the physical basis for a cosmological constant remained a mystery until the 1960's when it was realised that zero-point vacuum fluctuations must respect Lorenz invariance and therefore have the form $\langle T_{ik} \rangle = \Lambda g_{ik}$. As it turns out, the vacuum expectation value of the energy momentum is divergent both for bosonic and fermionic fields, and this gives rise to what is known as 'the cosmological constant problem'. Indeed the effective cosmological constant generated by vacuum fluctuations is

$$\frac{\Lambda}{8\pi G} = \langle T_{00} \rangle_{\rm vac} \propto \int_0^\infty \sqrt{k^2 + m^2} k^2 dk \ , \tag{4}$$

this integral diverging like k^4 implies an infinite value for the vacuum energy. Even if one chooses to 'regularise' $\langle T_{ik} \rangle$ by invoking an ultraviolet cutoff at the Planck scale, one is still left with an enormously large value for the vacuum energy $\langle T_{00} \rangle_{\rm vac} \simeq c^5/G^2\hbar \sim\sim 10^{76}{\rm GeV}^4$ which is 123 orders of magnitude larger than the currently observed $\rho_\Lambda \simeq 10^{-47}{\rm GeV}^4$. A smaller ultraviolet cut-off does not fare much better since a cutoff at the QCD scale results in $\Lambda^4_{QCD} \sim 10^{-3}{\rm GeV}^4$, which is still forty orders of magnitude larger than observed.

In the 1970's the discovery of supersymmetry led to the hope that, since bosons and fermions (of identical mass) contribute equally but with opposite sign to the vacuum expectation value of physical quantities, the cosmological constant problem may be resolved by a judicious balance between bosons and fermions in nature. However supersymmetry (if it exists) is broken at the low temperatures prevailing in the universe today and on this account one should expect the cosmological constant to vanish in the early universe, but to reappear during late times when the temperature has dropped below $T_{\rm SUSY}$. This is clearly an undesirable scenario and almost the very opposite of what one is looking for, since, a large value of Λ at an early time is useful from the viewpoint of inflation, whereas a very small current value of Λ is in agreement with observations [1, 7].

The cosmological constant is also relevant from the perspective of models with spontaneous symmetry breaking [8]. Indeed, if one examines the Lagrangian

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\, g^{ij}\partial_i\phi\partial_j\phi - V(\phi), \\ V(\phi) &= V_0 - \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4, \end{aligned} \tag{5}$$

one notices that the symmetric state at $\phi = 0$ is unstable and the system settles in the ground state $\phi = +\sigma$ or $\phi = -\sigma$, where $\sigma = \sqrt{\mu^2/\lambda}$, which breaks the reflection symmetry $\phi \leftrightarrow -\phi$ present in the Lagrangian. For $V_0 = 0$ this potential gives rise to a large negative cosmological constant $\Lambda_{eff} = V(\phi = \sigma) = -\mu^4/4\lambda$ in the broken symmetry state. This embarassing situation can be avoided only if one chooses a value for V_0 which almost exactly cancels Λ_{eff}, namely $V_0 \simeq +\mu^4/4\lambda$ so that $\Lambda_{eff}/8\pi G = V_0 - \mu^4/4\lambda \simeq 10^{-47}\mathrm{GeV}^4$.

The cosmological consequences of this rather ad-hoc 'regularisation' exercise are instructive. Unless the value of Λ_{eff} lies in a very small window, the universe will be a very different place from the one which we are used to. For instance a negative value of the Λ-term $\Lambda_{eff}/8\pi G < -10^{-43}\mathrm{GeV}^4$ will cause the universe to recollapse (the effect of Λ is attractive now instead of being repulsive) only a billion years or so after the big bang – a period which is much too short for galaxies to form and for life (as we know it) to emerge. On the other hand a large positive $\Lambda_{eff}/8\pi G > 10^{-43}\mathrm{GeV}^4$ makes the universe accelerate much before the present epoch, thereby inhibiting structure formation and precluding the emergence of life.

The very small window in Λ which allows life to emerge has led several prominent cosmologists to develop and propose anthropic arguments for the existence of a small cosmological constant [10, 9, 11]. One possibility is "if our big bang is just one of many big bangs, with a wide range of vacuum energies, then it is natural that some of these big bangs should have a vacuum energy in the narrow range where galaxies can form, and of course it is just these big bangs in which there could be astronomers and physicists wondering about the vacuum energy" [11].

I will not discuss the anthropic argument any further in these lectures but will point the interested reader to [10, 9, 11] for further discussion of this issue.

It is important to note that there is no known fundamental symmetry in nature which will set the value of Λ to zero. In its absence, the small value of the cosmological constant today remains somewhat of a dilemma which remains to be fully understood and resolved. [2]

2 Dark Energy model building

The cosmological constant is but one example of a form of matter (dark energy) which can drive an accelerated phase in the expansion history of our universe.

I now focus on the different approaches which have been used to fashion models of dark energy, their primary motivation being to allow that Universe to accelerate at the present (late) cosmological epoch.

[2] The important role played by symmetries is illustrated by the U(1) gauge symmetry of electrodynamics whose presence implies a zero rest mass for the photon. No analogous symmetry exists for neutrino's and indeed recent experiments do indicate that the neutrino could have a small mass.

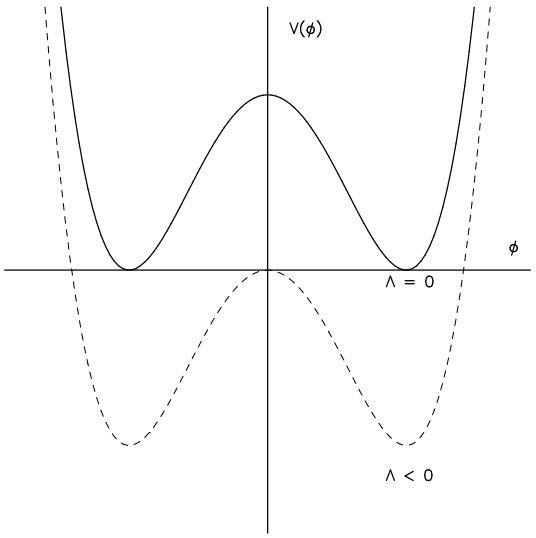

Fig. 1. Spontaneous symetry breaking in many field theory models takes the form of the Mexican top hat potential shown above. The dashed line shows the potential before the cosmological constant has been 'renormalized' and the solid line after. (From Sahni and Starobinsky 2000 [1].)

2.1 Dark energy with a constant equation of state (Quiessence)

For dark energy with a constant equation of state, equation (2) is easily generalised to

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_i (\rho_i + 3p_i) = -\frac{4\pi G}{3}\sum_i \rho_i(1+3w_i) \ , \tag{6}$$

where the summation is over all forms of matter present in the universe and $w = p/\rho$ is their equation of state. Eqn. (6) together with its companion equation

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i - \frac{k}{a^2} \tag{7}$$

completely describes the dynamics of a FRW universe (k/a^2 is the Gaussian curvature of space).

Clearly a universe consisting of only a single component will accelerate if $w < -1/3$.

Fluids satisfying $\rho + 3p \geq 0$ or $w \geq -1/3$ are said to satisfy the 'strong energy condition' (SEC). We therefore find that, in order to accelerate, 'dark energy' must violate the SEC. Another condition which is usually assumed to be sacrosanct, but has recently been called into question is the 'weak energy

condition' (WEC) according to which $\rho + p \geq 0$ or $w \geq -1$. Failure to satisfy the WEC can result in faster-than-exponential expansion for the universe and in a cosmic 'Big Rip', which we shall come to in a moment.

It is often more convenient to rewrite (6) in terms of the 'deceleration parameter'

$$q = -\frac{\ddot{a}}{aH^2} = \sum_i (\frac{8\pi G\rho_i}{3H^2})(1+3w_i)/2 = \frac{1}{2}\left(1 + 3w_{\rm DE}\Omega_{\rm DE}\right) \ , \tag{8}$$

where we have assumed a flat universe with $\Omega_{\rm m} + \Omega_{\rm DE} = 1$. The condition for accelerated expansion ($q < 0$) is therefore equivalent to

$$w < -\frac{1}{3(1-\Omega_{\rm m})} \ , \tag{9}$$

and for $\Omega_{\rm m} \simeq 1/3$ we get

$$w < -\frac{1}{2} \ . \tag{10}$$

If dark energy is described by an unevolving equation of state $w = p_{\rm DE}/\rho_{\rm DE}$, then the transition between deceleration and acceleration ($\ddot{a} = 0$) occurs at the redshift

$$(1+z_{\rm a})^{-3w} = -(1+3w)\frac{\Omega_{\rm DE}}{\Omega_m}. \tag{11}$$

Another important redshift describes the epoch when the densities in dark matter and dark energy are equal

$$(1+z_{\rm eq})^{3w} = \left(\frac{\Omega_m}{\Omega_{\rm DE}}\right), \ w < 0 \ . \tag{12}$$

Substituting $\Omega_\Lambda = 0.7, \Omega_m = 0.3$ we find $z_{\rm a} \simeq 0.73$, $z_{\rm eq} \simeq 0.37$ for LCDM cosmology. The fact that the acceleration of the universe is a fairly recent phenomenon illustrates the 'cosmic coincidence' puzzle according to which we appear to live during a special epoch when the densities in dark energy and in dark matter are almost equal. Dark enegy models with an unevolving equation of state need to have their initial conditions properly 'tuned' in order to dominate the universe at precisely the present epoch. This problem is most acute for the cosmological constant for which $\rho_\Lambda =$ constant whereas $\rho_m(t) \propto a^{-3}$ for matter and $\rho_r(t) \propto a^{-4}$ for radiation. Clearly, since the cosmological constant does not evolve (as its name suggests) while the densities of matter and radiation rapidly increase with redshift, it follows that to make $\rho_\Lambda = \Lambda/8\pi G \simeq 10^{-47}$ GeV^4 at the present time one requires $\rho_\Lambda/\rho_r \simeq 10^{-123}$ at the Planck time (when the temperature of the universe was $T \sim 10^{19}$ GeV), or $\rho_\Lambda/\rho_r \simeq 10^{-55}$ at the time of the electroweak phase transition ($T \sim 100$ GeV). Both cases demand an extreme fine-tuning in order to ensure that $\rho_\Lambda/\rho_r \sim 1$ today !

The fine tuning problem which plagues Λ also affects DE models in which w = constant. A combined analaysis of CMB, galaxy clustering and supernovae data indicates that a constant equation of state for dark energy must satisfy $w_{\rm DE} < -0.82$ at the 95% confidence level [12, 5], and it is easy to show that for these models the fine tuning (and cosmic coincidence) problems are almost as acute as they are for the cosmological constant.

Finally, eqn. (7) can be used to develop an expression for the Hubble parameter $H \equiv \dot{a}/a$ in terms of the cosmological redshift $z = a_0/a(t) - 1$:

$$H(z) = H_0 \left[\Omega_m (1+z)^3 + \Omega_{\rm DE}(1+z)^{3(1+w)}\right]^{1/2}, \tag{13}$$

where $H_0 = H(z=0)$ is the present value of the Hubble parameter, $\Omega_m = 8\pi G\rho_{0m}/3H_0^2$, $\Omega_{\rm DE} = 8\pi G\rho_{0\rm DE}/3H_0^2$, describe the dimensionless density of matter and dark energy respectively, ($w \equiv w_{\rm DE}$), and we have made the assumption of a flat universe so that $\Omega_m + \Omega_{\rm DE} = 1$.

In LCDM cosmology $w = -1$ and $\Omega_\Lambda = \Lambda/3H_0^2$, and the expansion factor has a simple and elegant form [1]

$$a(t) \propto \left(\sinh \frac{3}{2}\sqrt{\frac{\Lambda}{3}}ct\right)^{2/3}. \tag{14}$$

which smoothly interpolates between a matter dominated universe in the past ($a \propto t^{2/3}$) and accelerated expansion in the future ($a \propto \exp\sqrt{\Lambda/3}t$).

2.2 Quintessence

The fine tuning problem facing dark energy models with a constant equation of state can be alleviated if we assume that the equation of state is time dependent. An important class of models having this property are scalar fields which couple minimally to gravity and whose energy momentum tensor is

$$\rho \equiv T_0^0 = \frac{1}{2}\dot{\phi}^2 + V(\phi),\ P \equiv -T_\alpha^\alpha = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{15}$$

Potentials which are sufficiently steep in order to satisfy $\Gamma \equiv V''V/(V')^2 \geq 1$ have the interesting property that scalar fields approach a common evolutionary path from a wide range of initial conditions [13]. In these so-called 'tracker' models the scalar field density (and its equation of state) remains close to that of the dominant background matter during most of cosmological evolution.

An excellent example is provided by $V(\phi) = V_0/\phi^\alpha$ [14]. During tracking the ratio of the energy density of the scalar field (quintessence) to that of radiation/matter gradually increases $\rho_\phi/\rho_B \propto t^{4/(2+\alpha)}$ while its equation of state remains marginally smaller than the background value $w_\phi = (\alpha w_B - 2)/(\alpha + 2)$. For large values of ϕ this potential becomes flat and this ensures

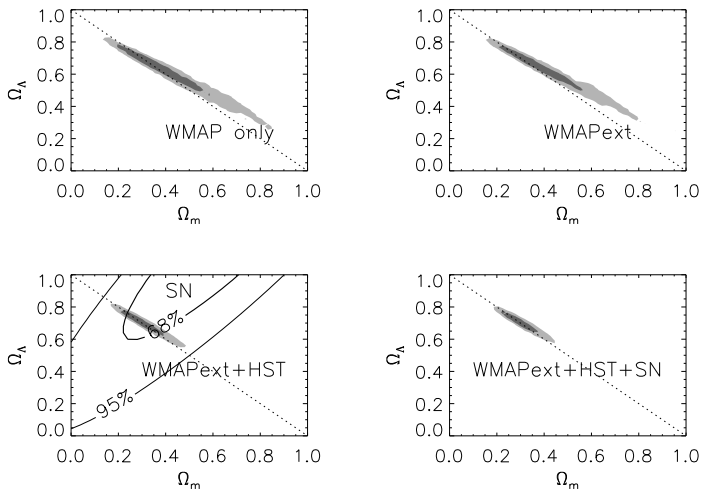

Fig. 2. Constraints on the density of dark matter Ω_m and dark energy in the form of a cosmological constant Ω_Λ, determined using WMAP (upper left), WMAP + other CMB experiments (WMAPext; upper right), WMAPext + HST key project data (lower left) and WMAPext + HST + supernova data (lower right); from Spergel et al (2003) [5].

that the scalar field rolls slower ($\dot{\phi}^2 \ll V(\phi)$) thereby allowing the universe to accelerate. Note that for quintessence fields the condition's (10) & (10) translate into

$$w_\phi < -\frac{1}{3} \;\Rightarrow\; \dot{\phi}^2 < V(\phi)$$
$$w_\phi < -\frac{1}{2} \;\Rightarrow\; \dot{\phi}^2 < \frac{2}{3}V(\phi)\ . \tag{16}$$

The scalar field therefore eventually dominates the density of the universe and generates a late-time epoch of accelerated expansion. (Current observations place the strong constrain $\alpha < 2$.)

A an extreme example of quintessence is provided by the exponential potential $V(\phi) = V_0 \exp\left(-\sqrt{8\pi}\lambda\phi/M_{\rm Pl}\right)$ [14, 15] for which

$$\frac{\rho_\phi}{\rho_B + \rho_\phi} = \frac{3(1+w_B)}{\lambda^2} = \text{constant} < 0.2, \tag{17}$$

ρ_B is the background energy density while w_B is the associated equation of state. The lower limit $\rho_\phi/\rho_{\rm total} < 0.2$ arises because of nucleosynthesis constraints which prevent the energy density in quintessence from being large initially (at $t \sim few$ sec.). Equation (17) suggests that the exponential potential will remain subdominant if it was so initially. Since the ratio $\rho_\phi/\rho_{\rm total}$

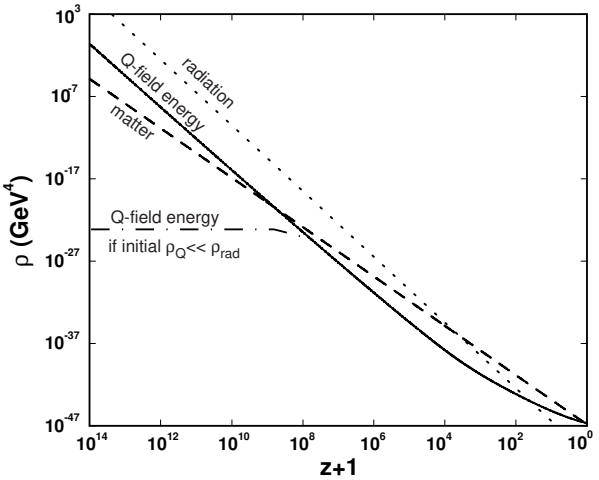

Fig. 3. The quintessence Q-field while rolling an inverse power law potential tracks first radiation then matter, before coming to dominate the energy density of the universe at present. If the initial value of the Q-field density is small then $\rho_{\rm Q}$ remains constant until $\rho_{\rm Q} \sim \rho_{\rm rad}$, and then follows the tracker. From Zlatev, Wang and Steinhardt [13].

remains fixed, exponential potentials on their own cannot supply us with a means of generating dark energy/quintessence at the present epoch for generic initial conditions. An interesting potential which interpolates between an exponential and a power law can however give rise to late time acceleration from tracker-like initial conditions [16]

$$V(\phi) = V_0[\cosh \lambda\phi - 1]^p, \tag{18}$$

has the property that $w_\phi \simeq w_B$ at early times whereas $\langle w_\phi \rangle = (p-1)/(p+1)$ at late times. Consequently (18) describes *quintessence* for $p \leq 1/2$ and pressureless 'cold' dark matter (CDM) for $p = 1$. Thus the cosine hyperbolic potential (18) is able to describe both dark matter and dark energy within a common framework. It is also important to emphasise that in the case of the potential (18) the scalar field *oscillates* about its zero value leading to an *averaged* equation of state which is negative (and which thereby causes the universe to accelerate). This model is therefore complementary to the inverse power law models discussed earlier, in which the scalar ϕ must roll to sufficiently large values ($\phi/m_P \gtrsim 1$) before the slow roll conditions are satisfied and the scalar field acts can cause the universe to accelerate. Some scalar field based dark energy models (quintessence) are listed in table 1.

Quintessence models can be divided into two categories: models which roll to large values of $\phi/m_P \gtrsim 1$ and models for which $\phi/m_P \ll 1$ at the

Quintessence Potential	Reference
$V_0 \exp(-\lambda\phi)$	Ratra & Peebles (1988), Wetterich (1988), Ferreira & Joyce (1998)
$m^2\phi^2, \lambda\phi^4$	Frieman et al (1995)
$V_0/\phi^\alpha, \alpha > 0$	Ratra & Peebles (1988)
$V_0 \exp(\lambda\phi^2)/\phi^\alpha, \alpha > 0$	Brax & Martin (1999,2000)
$V_0(\cosh\lambda\phi - 1)^p$,	Sahni & Wang (2000)
$V_0 \sinh^{-\alpha}(\lambda\phi)$,	Sahni & Starobinsky (2000), Ureña-López & Matos (2000)
$V_0(e^{\alpha\kappa\phi} + e^{\beta\kappa\phi})$	Barreiro, Copeland & Nunes (2000)
$V_0(\exp M_p/\phi - 1)$,	Zlatev, Wang & Steinhardt (1999)
$V_0[(\phi - B)^\alpha + A]e^{-\lambda\phi}$,	Albrecht & Skordis (2000)

Table 1.

present epoch. An important concern for the former is the effect of quantum corrections which, if large, could alter the shape of the quintessence potential [17, 18]. An important related issue is that the coupling between standard model fields and quintessence must be small in order to have evaded detection. Moreover it is interesting that even small couplings between quintessence and standard model fields can give rise to interesting changes in cosmology as discussed in [19].

I should emphasise that the shortage of time does not allow me to dwell on *all* models of dark energy discussed in the literature. (A number that is growing rapidly !) For this reason these lectures will not discuss DE due to vacuum polarization[20, 21], k-essence [22], models with non-minimally coupled scalars or models which either try to unify or consider explicit interaction terms between dark energy and dark matter, Cardassian expansion [23], Quasi-Steady State Cosmology [24], scalar-tensor models [25, 26, 27, 28, 29, 30, 31, 32, 33]; see also [34, 35, 36, 37, 38, 39].

The reader is referred to the reviews [1, 40, 41, 7, 42, 2] for other coverage of theoretical models of DE.

2.3 Chaplygin gas

An interesting and very different model of dark energy is the Chaplygin gas [43]. Chaplygin gas possesses the equation of state

$$p_c = -A/\rho_c \ , \tag{19}$$

so that its energy density evolves according to

$$\rho_c = \sqrt{A + B(1+z)^6} \ , \tag{20}$$

which implies $\rho_c \to \sqrt{A}$ as $z \to -1$ and $\rho_c \to \sqrt{B}(1+z)^3$ as $z \gg 1$. Remarkably, the Chaplygin gas appears to behave like pressureless dust at early times and like a cosmological constant during very late times ! The Hubble parameter for a universe containing cold dark matter and the Chaplygin gas is given by

$$H(z) = H_0 \left[\Omega(1+z)^3 + \frac{\Omega}{\kappa}\sqrt{\frac{A}{B} + (1+z)^6} \right]^{1/2} \ , \tag{21}$$

where $\kappa = \rho_{0m}/\sqrt{B}$. It is easy to see from (21) that

$$\kappa = \frac{\rho_{0m}}{\rho_c}(z \to \infty) \ . \tag{22}$$

Thus, κ defines the ratio between CDM and the Chaplygin gas energy densities at the commencement of the matter-dominated stage. It is easy to show that

$$A = B\left\{ \kappa^2 \left(\frac{1-\Omega_m}{\Omega_m} \right)^2 - 1 \right\} \ . \tag{23}$$

It should be mentioned that the Chaplygin gas can be derived from the Born-Infeld form of the lagrangian density

$$\mathcal{L} = -V_0\sqrt{1 - \phi_{,\mu}\phi^{,\mu}} \ , \tag{24}$$

where $\phi_{,\mu} \equiv \partial\phi/\partial x^\mu$. For time-like $\phi_{,\mu}$ one can define a four velocity

$$u^\mu = \frac{\phi^{,\mu}}{\sqrt{\phi_{,\alpha}\phi^{,\alpha}}} \ , \tag{25}$$

this leads to the standard form for the hydrodynamical energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu} \ , \tag{26}$$

where [44, 45]

$$\rho = \frac{V_0}{\sqrt{1 - \phi_{,\mu}\phi^{,\mu}}} \ , \quad p = -V_0\sqrt{1 - \phi_{,\mu}\phi^{,\mu}} \ , \tag{27}$$

i.e. we have recovered (19) with $A = V_0^2$.

A 'generalized Chaplygin gas' has also been proposed for which $p \propto -1/\rho^{\alpha}$. The equation of state in this case is

$$w(a) = -\frac{|w_0|}{\left[|w_0| + \frac{1-|w_0|}{a^{3(1+\alpha)}}\right]} \ , \tag{28}$$

which interpolates between $w = 0$ at early times ($a \ll 1$) and $w = -1$ at very late times ($a \gg 1$). w_0 is the current equation of state when $a = 1$. (The constant α regulates the transition time in the equation of state.) WMAP, supernovae and large scale sructure data have all been used to test Chaplygin gas models; see for instance [46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56].

2.4 Phantom Dark Energy

Caldwell was probably the first person to notice that a very good fit to the supernova-derived luminosity distance was provided by dark energy which violated the WEC so that $w < -1$. He called this Phantom dark energy [57]. Indeed, a study of high-z Sn by [58] finds that the DE equation of state has a 99% probability of being < -1 if no priors are placed on Ω_m !

A universe filled with Phantom energy has some interesting if bizarre properties.

- If $t_{\rm eq}$ marks the epoch when the densities in matter and phantom energy are equal then expansion factor of a universe dominated by phantom energy grows as

$$a(t) \simeq a\left(t_{\rm eq}\right)\left[(1+w)\frac{t}{t_{\rm eq}} - w\right]^{2\Big/3(1+w)} \quad , \quad w < -1 \ , \tag{29}$$

  and therefore *diverges in a finite* amount of cosmic time

$$a(t) \to \infty \ \ \text{as} \ \ t \to t_{\rm BR} = \left(\frac{w}{1+w}\right) t_{\rm eq} \, . \tag{30}$$

  Indeed if we substitute $w < -1$ in (13) we immediately find that Hubble parameter also diverges as $t \to t_{\rm BR}$, implying that an infinitely rapid expansion rate for the universe has been reached in a *finite* time. The divergence of the Hubble parameter is associated with the divergence of phantom density which grows without bound

$$\rho(t) \propto \left[(1+w)\frac{t}{t_{\rm eq}} - w\right]^{-2} , \tag{31}$$

  and reaches a singular value in a finite interval of time $\rho(t) \to \infty$, $t \to t_{\rm BR}$. Thus a universe dominated by Phantom energy culminates in a future curvature singularity ('Big Rip') at which the notion of a classical spacetime breaks down. (See also [59, 57, 60, 61, 62, 63, 64, 65, 66].)

- A negative equation of state suggests that the effective velocity of sound in the medium $v = \sqrt{|dp/d\rho|}$ can become larger than the velocity of light.
- Although a dynamical model of phantom energy can be constructed with the 'wrong' sign of the kinetic term, see (15), such models are plagued with instabilities at the quantum level [67] which makes their existence highly suspect.
  It should be pointed out that phantom is by no means the only way to get $w < -1$. A model with similar properties but sharing none of phantom's pathologies is the braneworld model of [68, 69] which has $w_{\rm eff} < -1$ today but does not run into a 'Big Rip' in the future. We shall discuss this model in the next section.

3 Braneworld models of Dark Energy

The main approaches to Dark Energy model building discussed earlier relied on changing the RHS of the Einstein field equations (1). However, it might well be that the riddle of Dark Energy is associated with the LHS ! (Indeed Eintein's original 'discovery' of Λ, placed the cosmological constant in the LHS of his field equations (1).) In this talk I will focus on one such approach which rests on the notion that space-time is higher-dimensional, and that our observable universe is a (3+1)-dimensional 'brane' which is embedded in a (4+1)-dimensional 'bulk' space-time [3]. As we shall see, higher-dimensional braneworld models allow the expansion dynamics to be radically different from that predicted by conventional Einstein gravity in 3+1 dimensions. Some cosmological 'surpises' which spring from Braneworld models include:

- Both early and late time acceleration can be successfully unified within a single scheme (Quintessential Inflation) in which the very same scalar field which drives Inflation at early times becomes Quintessence at late times.
- The (effective) equation of state of dark energy in the braneworld scenario can be 'phantom-like' ($w < -1$) or 'Quintessence-like' ($w > -1$). (These two possibilities are essentially related to the two distinct ways in which the brane can be embedded in the bulk.)
- The acceleration of the universe can be a *transient* phenomenon: braneworld models accelerate during the present epoch but return to matter-dominated expansion at late times.
- A class of braneworld models encounter a *Quiescent future singularity*, at which $\dot{a} \to$ constant, but $\ddot{a} \to -\infty$. The surprising feature of this singularity is that while the Hubble parameter, density and pressure remain finite, the deceleration parameter and all curvature invariants *diverge* as the singularity is approached.

[3] The subsequent discussion is primarily based on [3] to which the reader is referred for more details.

- A spatially flat Braneworld can *mimick* a closed universe and *loiter* at large redshifts.
- A braneworld embedded in a five-dimensional space in which the extra (bulk) dimension is time-like can *bounce at early times*, thereby generically avoiding the big bang singularity. Cyclic models of the universe with successive expansion-contraction cycles can be constructed based on such a bouncing braneworld.

Let us now dwell a little on some of these cosmological properties of braneworld models.

3.1 Quintessential Inflation on the Brane

An intriguing issue in cosmology is that the universe appears to accelerate twice: once at the very beginning during Inflation and again about 10 billion years later, during the present epoch. Although most theoretical models assume that there is no real connection between the two epochs and that Inflation and Dark energy are distinct physical entities, it might well be that the two phenomena are in fact related, and that the same scalar field which initially drives inflation, later, when its density has been considerably reduced, plays the role of Quintessence. This notion of Quintessential Inflation was first explored in the context of the Einstein gravity in 3+1 dimensions by Peebles and Vilenkin [70]. The possibility that Braneworld models could provide a more efficient realisation of this scenario was suggested by Copeland, Liddle and Lidsey [71] and subsequently discussed in greater detail by several authors [72, 73, 76, 74, 75].

The main line of reasoning behind this approach is simple. In the Randall–Sundrum model [78] the modified Einstein equations on the brane contain high-energy corrections as well as the projection of the Weyl tensor from the bulk on to the brane. The Friedmann equation on the brane in this case becomes [79]

$$H^2 + \frac{\kappa}{a^2} = \frac{\Lambda_{\rm b}}{6} + \frac{C}{a^4} + \frac{(\rho+\sigma)^2}{9M^6}\,. \tag{32}$$

where $\kappa = 0, \pm 1$, σ is the brane tension, $\Lambda_{\rm b}$ is the value of the cosmological constant in the five-dimensional 'bulk' and C/a^4 is the 'dark radiation' term which describes the projected five-dimensional degrees of freedom onto the brane. It is easy to see from (32) that

$$H^2(a \to \infty) = \frac{\Lambda_{\rm RS}}{3}\ , \tag{33}$$

where

$$\Lambda_{\rm RS} = \frac{\Lambda_{\rm b}}{2} + \frac{\sigma^2}{3M^6} \tag{34}$$

is the effective four dimensional cosmological constant in the Randall–Sundrum model.

During Inflation the expansion of the universe is driven by a scalar field propagating on the brane with energy density and pressure given by

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi), \qquad p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi) \ , \tag{35}$$

while the evolution equation for the scalar field is

$$\ddot{\phi}(t) + 3H\dot{\phi}(t) + V_{,\phi} = 0 \ . \tag{36}$$

Braneworld models add an interesting new dimension to the scalar-field dynamics. As demonstrated by (32), for $\rho \gg \sigma$ a quadratic density term appears in the modified Friedmann equation on the brane. This radically alters the expansion dynamics at early epochs by speeding up the rate of expansion and the value of the Hubble parameter, which now becomes $H \propto \rho$ instead of the more conventional $H \propto \sqrt{\rho}$. Since the value of H affects the motion of ϕ through the damping term '$3H\dot{\phi}$' in (36), the scalar field experiences significantly greater damping in braneworld cosmology than in conventional GR.

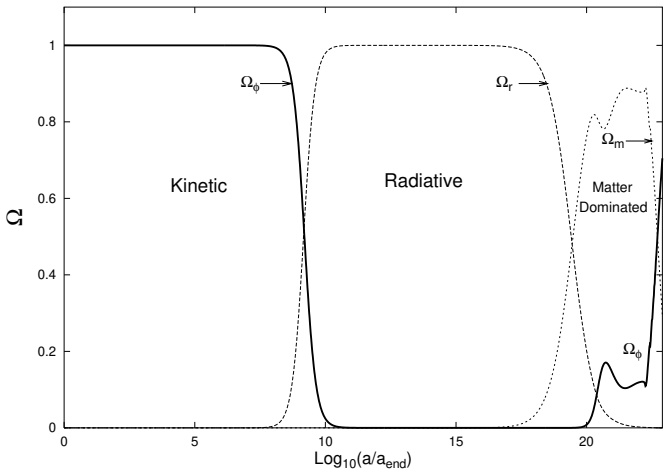

Fig. 4. The post-inflationary density parameter Ω is plotted for the scalar field (solid line), radiation (dashed line), and cold dark matter (dotted line) in the quintessential-inflationary model described by the potential [16] $V(\phi) = V_0[\cosh \lambda\phi - 1]^p$ with $p = 0.2$. Late-time oscillations of the scalar field ensure that the mean equation of state turns negative $\langle w_\phi \rangle \simeq -2/3$, giving rise to the current epoch of cosmic acceleration with $a(t) \propto t^2$ and present-day values $\Omega_{0\phi} \simeq 0.7$, $\Omega_{0m} \simeq 0.3$. Note the existence of two epochs, early and late, when the scalar field dominates the matter density of the universe. Figure courtesy of [75].

Consequently, inflation on the brane can be realized by very steep potentials – precisely those that are used to describe Quintessence. The braneworld scenario therefore provides us with the opportunity to unify inflation and dark energy through the notion of quintessential inflation [70, 77, 71, 72, 73, 76, 75]. An example of quintessential inflation is shown in Figure 4.

An important property of Quintessential Inflation on the brane is that once the inflationary regime is over and braneworld corrections cease to play an important role, the extreme steepness of the potential causes the scalar field to plunge down its potential resulting in a 'kinetic regime' prior to reheating during which $p_\phi \simeq \rho_\phi$. It is well known that the spectrum of relic gravity waves created quantum mechanicaly during Inflation is sensitive to and bears an imprint of the post-inflationary equation of state [80, 81]. The effect of the kinetic regime is to create a 'blue' gravity wave spectrum on short wavelength scales which is an important observational signature of Quintessential Inflation on the brane [73, 75]; see also [82].

3.2 Braneworld dark energy with $w \leq -1$ and $w \geq -1$

A radically different way of making the universe accelerate was suggested in [83, 68]. The braneworld model which I now discuss presents a successful synthesis of the higher-dimensional ansatzes proposed by Randall and Sundrum [78] and Dvali, Gabadadze, and Porrati [83], and is described by the action [84]

$$S = M^3 \left[\int_{\rm bulk} (\mathcal{R} - 2\Lambda_{\rm b}) - 2\int_{\rm brane} K\right] + \int_{\rm brane} \left(m^2 R - 2\sigma\right) + \int_{\rm brane} L(h_{ab}, \phi)\,. \tag{37}$$

Here, $\mathcal{R}$ is the scalar curvature of the five-dimensional metric g_{ab} in the bulk, and R is the scalar curvature of the induced metric $h_{ab} = g_{ab} - n_a n_b$ on the brane, where n^a is the vector field of the inner unit normal to the brane. The quantity $K = K_{ab}h^{ab}$ is the trace of the symmetric tensor of extrinsic curvature $K_{ab} = h^c{}_a \nabla_c n_b$ of the brane, and $L(h_{ab}, \phi)$ denotes the Lagrangian density of the four-dimensional matter fields ϕ whose dynamics is restricted to the brane (the notation and conventions are those of [85]). Integrations over the bulk and brane are taken with the natural volume elements $\sqrt{-g}\, d^5x$ and $\sqrt{-h}\, d^4x$, respectively. The constants M and m denote, respectively, the five-dimensional and four-dimensional Planck masses, $\Lambda_{\rm b}$ is the five-dimensional (bulk) cosmological constant, and σ is the brane tension.

An important difference between the action (37) and that describing the Randall–Sundrum cosmology is the presence of the term $m^2 \int R d^4x$ in (37). This term can be thought of as resulting from the backreaction of quantum fluctuations of matter fields residing on the brane, and this mechanism of generating the gravitational part of the action was proposed by Sakharov in a seminal paper in 1967 [86]. The effect of including such a term in the braneworld action was explored in [84, 83, 68].

Action (37) leads to the following expression for the Hubble parameter on the brane for a *spatially flat* universe [68]:

$$H^2(a) = \frac{A}{a^3} + B + \frac{2}{\ell^2}\left[1 \pm \sqrt{1 + \ell^2\left(\frac{A}{a^3} + B - \frac{\Lambda_{\rm b}}{6} - \frac{C}{a^4}\right)}\right], \qquad (38)$$

where

$$A = \frac{\rho_0 a_0^3}{3m^2}, \quad B = \frac{\sigma}{3m^2}, \quad \ell = \frac{2m^2}{M^3}. \qquad (39)$$

Note that the four-dimensional Planck mass m is related to the effective Newton's constant on the brane as $m = 1/\sqrt{8\pi G}$. The two Planck masses M and m define a new length scale $\ell = 2m^2/M^3 \sim cH_0^{-1}$ in a braneworld which begins to accelerate at the current epoch [83, 68]. (Other applications of braneworlds to the late-time acceleration of the universe are discussed in [87, 88, 89, 90].)

The two signs in (38) correspond to two branches of braneworld models and are related to the two different ways in which the brane can be embedded in the bulk. As shown in [68], the '+' sign in (38) corresponds to late time acceleration of the universe driven by dark energy with an 'effective' equation of state $w \geq -1$ (we call this model BRANE2) whereas the '−' sign is associated with phantom-like behaviour $w \leq -1$ (this model is called BRANE1).

Some limiting cases of (38) will be of interest to the reader:

1. For $m = 0$ equation (38) reduces to a Randall–Sundrum universe [79] described earlier in (32).
2. In the other extreme case when $M = 0$, extra-dimensional effects become unimportant, and (38) reduces to the LCDM model

$$H^2(a) = \frac{A}{a^3} + B. \qquad (40)$$

3. Finally, when $\Lambda_{\rm b} = 0$ and $\sigma = 0$, (38) reduces to the DGP braneworld [83].

Consider now the Braneworld with the '$-$' sign on the right-hand side of (38), *i.e.* BRANE1. In this case the Hubble parameter on the brane can be rewritten as:

$$H^2(a) = \frac{A}{a^3} + \Lambda_{\rm eff}\,.$$

The term $\Lambda_{\rm eff}$ is composed of two terms, namely, a constant Λ-term and a 'screening term':

$$\Lambda_{\rm eff} = \underbrace{(B + \frac{2}{\ell^2})}_{} - \underbrace{\frac{2}{\ell^2}\sqrt{1 + \ell^2\left(\frac{A}{a^3} + B\right)}}_{} \qquad (41)$$

$$\Downarrow \qquad\qquad \Downarrow \qquad (42)$$

$$\Lambda \qquad \text{Screening term}$$

Since the screening term decreases with time, the value of **the effective cosmological constant $\Lambda_{\rm eff}$ increases**. Therefore the Braneworld behaves just like a Phantom model ($w < -1$) but without Phantom's problems: no violation of the weak energy condition and no future singularity. Indeed, from (41) its clear that the universe evolves to ΛCDM in the future (see also [88, 59, 91, 57, 67, 92, 60, 61, 62] for discussions of related issues).

The fact that the Braneworld model (38) can give rise to Phantom-like behaviour can also be seen if we rewrite (38) in terms of the cosmological redshift 'z' so that [68]

$$\frac{H^2(z)}{H_0^2} = \Omega_{\rm m}(1+z)^3 + \Omega_\sigma + 2\Omega_l \pm 2\sqrt{\Omega_l}\,\sqrt{\Omega_{\rm m}(1+z)^3 + \Omega_\sigma + \Omega_\ell + \Omega_{\Lambda_{\rm b}}}\,, \qquad (43)$$

where we have set the dark radiation term in (38) to zero ($C = 0$) and

$$\Omega_\ell = \frac{1}{\ell^2 H_0^2}\ ,\ \Omega_{\rm m} = \frac{\rho_{0m}}{3m^2H_0^2}\ ,\ \Omega_\sigma = \frac{\sigma}{3m^2H_0^2}\ ,\ \Omega_{\Lambda_{\rm b}} = -\frac{\Lambda_{\rm b}}{6H_0^2}\ . \qquad (44)$$

The quantities Ω_i satisfy the constraint equation

$$\Omega_{\rm m} + \Omega_\kappa + \Omega_\sigma \pm 2\sqrt{\Omega_\ell}\,\sqrt{1 - \Omega_\kappa + \Omega_{\Lambda_{\rm b}}} = 1\,, \qquad (45)$$

where the $\pm$ sign in (45) is commensurate with that in (43). For future reference we also note the cosmological density associated with the four-dimensional cosmological constant (34) in Randall–Sundram cosmology

$$\Omega_{\rm RS} = \frac{\Lambda_{\rm RS}}{3H_0^2} = \frac{\Omega_\sigma^2}{4\Omega_\ell} - \Omega_{\Lambda_{\rm b}}\,. \qquad (46)$$

The expression for the current value of the effective equation of state is

$$w_0 = \frac{2q_0 - 1}{3\,(1 - \Omega_{\rm m})} = -1 \pm \frac{\Omega_{\rm m}}{1 - \Omega_{\rm m}} \sqrt{\frac{\Omega_\ell}{\Omega_{\rm m} + \Omega_\sigma + \Omega_\ell + \Omega_{\Lambda_{\rm b}}}}\,, \qquad (47)$$

and we immediately find that $w_0 \leq -1$ when we take the lower sign in (47), which corresponds to choosing one of two possible embeddings of this braneworld in the higher dimensional bulk (this is BRANE1). (The second choice of embedding (BRANE2) gives $w_0 \geq -1$.) It is important to note that all BRANE1 models have $w_0 \leq -1$ and $w(z) \simeq -0.5$ at $z \gg 1$. They therefore successfully cross the 'great divide' at $w = -1$ (see page 26 of [68] for a discussion of this issue).

3.3 Transient Dark Energy

Braneworld cosmology has the interesting property that it allows the acceleration of the universe to be a transient phenomenon. (In [68] these models were referred to as 'Disappearing Dark Energy' (DDE).) Indeed, by setting the effective four dimensional cosmological constant (34) to zero so that $\Omega_{\rm RS} = 0$ in (46) we get

$$\Omega_\sigma = \pm 2\sqrt{\Omega_\ell \Omega_{\Lambda_{\rm b}}}\,. \qquad (48)$$

Selecting the lower sign in (48) and substituting $\Omega_\sigma = -2\sqrt{\Omega_\ell \Omega_{\Lambda_{\rm b}}}$ in (43) we find that[4]

$$\left[\frac{H(z=-1)}{H_0}\right]^2 = 2\sqrt{\Omega_\ell}\left(\sqrt{\Omega_\ell} - \sqrt{\Omega_{\Lambda_{\rm b}}} \pm \left|\sqrt{\Omega_\ell} - \sqrt{\Omega_{\Lambda_{\rm b}}}\right|\right)\,. \qquad (49)$$

which *vanishes* for the upper sign when $\Omega_\ell \leq \Omega_{\Lambda_{\rm b}}$. This implies that, since the effective four dimensional cosmological constant vanishes, the universe reverts to matter dominated expansion in the future:
$H^2(z \to -1) \simeq \Omega_m (1+z)^3 \to 0$. The current acceleration of the universe is therefore a transient phase in this model, as illustrated in figure 5.

It is well known that an eternally accelerating universe is endowed with a cosmological event horizon which prevents the construction of a conventional S-matrix describing particle interactions within the framework of string or M-theory [93]. A transiently accelerating braneworld may therefore help reconcile string/M-theory with accelerating cosmology, which is an attractive feature of this braneworld cosmology.

3.4 Quiescent Future Singularity

As remarked earlier, an interesting property of the Braneworld (38) is that the late-time expansion of the universe can culminate in a 'Quiescent' singularity at which the density, pressure and Hubble parameter remain finite

[4] Note that the case $\Omega_\sigma = 2\sqrt{\Omega_\ell \Omega_{\Lambda_{\rm b}}}$ leads to transient acceleration with $\Omega_m > 1$ which is unrealistic [68].

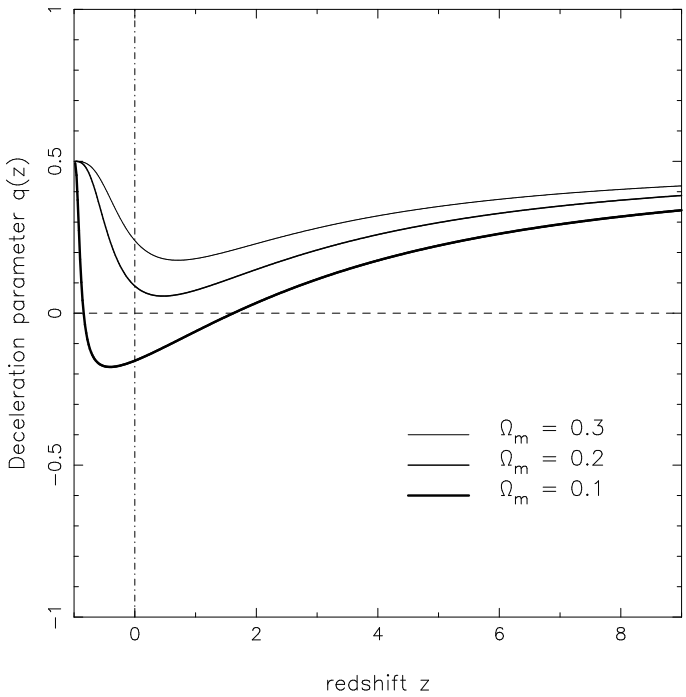

Fig. 5. The deceleration parameter $q = -\ddot{a}/aH^2$ is plotted as a function of redshift for the BRANE2 model with the Randall–Sundrum constraint $\Omega_\sigma = -2\sqrt{\Omega_\ell\,\Omega_{\Lambda_{\rm b}}}$. The vertical (dot-dashed) line at $z = 0$ marks the present epoch, while the horizontal (dashed) line at $q = 0$ corresponds to a Milne universe $[a(t) \propto t]$ which neither accelerates nor decelerates. The case $q = 0.5$ corresponds to matter dominated expansion. Note that the universe ceases to accelerate and becomes matter dominated in the past *as well as in the future*. Figure courtesy of [68].

while the deceleration parameter and geometrical invariants constructed from the Riemann tensor diverge [94].

In order to appreciate the origin of such unusual future singularities in the braneworld, consider again the expansion law (43). For simplicity, we shall only discuss the solution corresponding to the '+' sign in (43) (called BRANE2 in [68]).

A necessary and sufficient condition for the existence of a Quiescent singularity is for the inequality

$$\Omega_\sigma + \Omega_\ell + \Omega_{\Lambda_{\rm b}} < 0\,, \tag{50}$$

to be satisfied [94]. In this case the expression under the square root of (43) becomes zero at a suitably late time, and the cosmological solution *cannot be extended beyond this time*. It can be shown that the scale factor $a(t)$ and its first time derivative remain finite, while all the higher time derivatives of $a(t)$ *tend to infinity* as the singularity is approached.

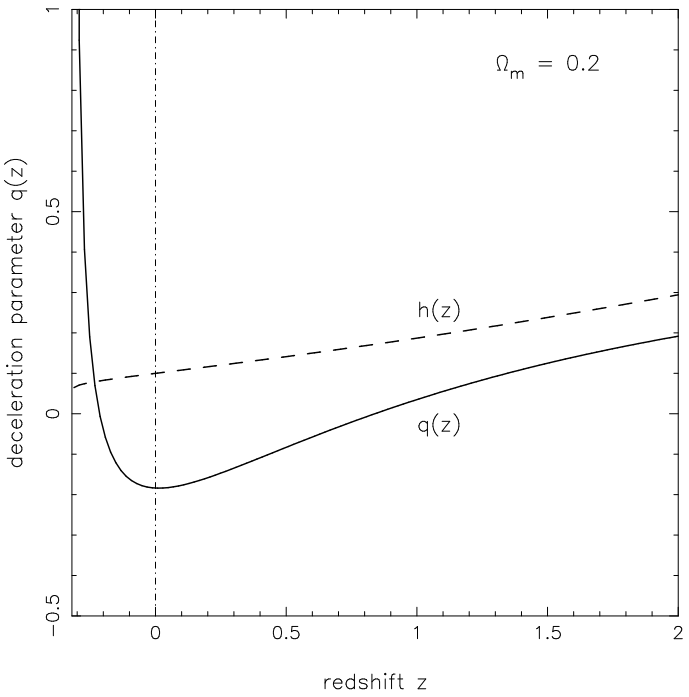

Fig. 6. The deceleration parameter (solid line) is shown for a braneworld model with $\Omega_{\rm m} = 0.2$, $\Omega_\ell = 0.4$, $\Omega_\Lambda = \Omega_\kappa = 0$, and Ω_σ determined from (45). We find that $q(z) \to 0.5$ for $z \gg 1$ while $q(z) \to \infty$ as $z \to -0.312779...$ Currently $q_0 < 0$, which indicates that the universe is accelerating. Also shown is the dimensionless Hubble parameter $h(z) = 0.1 \times H(z)/H_0$ (dashed line) which remains finite in this model. The vertical line at $z = 0$ shows the present epoch. Figure courtesy of [94].

The limiting redshift, $z_s = a_0/a(z_s) - 1$, at which the braneworld becomes singular is given by

$$z_s = \left(-\frac{\Omega_\sigma + \Omega_\ell + \Omega_{\Lambda_{\rm b}}}{\Omega_{\rm m}} \right)^{1/3} - 1 \,. \tag{51}$$

The time of occurance of the singularity (measured from the present moment) can easily be determined from

$$T_s = t(z = z_s) - t(z = 0) = \int_{z_s}^{0} \frac{dz}{(1+z)H(z)}, \tag{52}$$

where $H(z)$ is given by (43). In Fig. 6 we show a specific braneworld model having $\Omega_{\rm m} = 0.2$, $\Omega_\ell = 0.4$, $\Omega_{\Lambda_{\rm b}} = \Omega_\kappa = 0$. In keeping with observations of high redshift supernovae our model universe is currently accelerating [4], but will become singular at $z_s \simeq -0.3 \Rightarrow a(z_s) \simeq 1.4 \times a_0$, i.e. after $T_s \simeq$

4.5 h_{100}^{-1} Gyr ($h_{100} = H_0/100$ km/sec/Mpc). Figure 6 demonstrates that the deceleration parameter becomes singular as z_s is approached:

$$q = -\frac{\ddot{a}}{aH^2} \equiv \frac{H'}{H}(1+z) - 1\,; \quad \lim_{z \to z_s} q(z) \to \infty\,, \tag{53}$$

while the Hubble parameter remains finite:

$$\frac{H^2(z_s)}{H_0^2} = \Omega_\ell - \Omega_{\Lambda_{\rm b}}\,. \tag{54}$$

The possible presence of such 'Quiescent' singularity in Braneworld theory can also be seen from the following consideration. As shown in [68], the action (37) leads to the bulk being described by the usual Einstein equation with cosmological constant:

$$\mathcal{G}_{ab} + \Lambda_{\rm b} g_{ab} = 0\,, \tag{55}$$

while the field equation on the brane is

$$m^2 G_{ab} + \sigma h_{ab} = \tau_{ab} + \underline{M^3\,(K_{ab} - h_{ab}K)}\,. \tag{56}$$

Here, τ_{ab} is the stress-energy tensor on the brane. A divergent form for the extrinsic curvature (underlined term) caused by a singular embedding of the brane in the bulk will, through (56), lead to a singular value for the Einstein tensor G_{ab} even though all components of the stress-energy tensor on the brane remain finite!

More generally it is well known that Quiescent singularites occur when the original equations of motion are non-linear with respect to the highest derivative. They have earlier been discussed in the context of Einstein gravity with the conformal anomaly [95]. (This result is not surprising in view of the formal similarity between braneworld theory and GR-based models with the conformal anomaly, discussed in [68].) 'Determinant singularities' having a similar structure and properties are known to arise in the anisotropic Bianchi I model containing a dilaton coupled to a Gauss–Bonnet term in the action and also in other cosmological models [96, 97]. For instance [96] refer to singularities in which the deceleration parameter tends to infinity as the 'Big Brake' while in [97] they are called 'sudden' singularities.

We should also emphasise that both the past and future quiescent singularities occur for a wide range in parameter space and might provide an interesting alternative to the 'big bang'/'big crunch' singularities of general relativity.

3.5 Bouncing Braneworlds

A remarkable feature of Braneworld cosmology is that the expansion dynamics of the early universe can be *non-singular*. Indeed, as shown in [98] bouncing Braneworld models can be constructed, based upon a Randall–Sundrum type

action, but in which the extra dimension is *time-like*. In this case one starts from the action [98]

$$S = M^3 \left[\int_{\rm bulk} (\mathcal{R} - 2\Lambda) - 2\epsilon \int_{\rm brane} K \right] - \int_{\rm brane} 2\sigma + \int_{\rm brane} L(h_{ab}, \phi) \,. \quad (57)$$

where the parameter $\epsilon = 1$ if the signature of the bulk space is Lorentzian, so that the extra dimension is spacelike, and $\epsilon = -1$ if the signature is $(-,-,+,+,+)$, so that the extra dimension is timelike. The evolution equations resulting from (57) have the form [98]

$$H^2 + \frac{\kappa}{a^2} = \frac{\Lambda_{\rm eff}}{3} + \frac{8\pi G_{\rm N}\rho}{3} + \frac{\epsilon\rho^2}{9M^6} + \frac{C}{a^4} \,. \quad (58)$$

Where $G_{\rm N} = \epsilon\sigma/12\pi M^6$ is the effective gravitational constant, and $\Lambda_{\rm eff} = \Lambda/2 + \epsilon\sigma^2/3M^6$ is the effective cosmological constant on the brane. For $\epsilon = 1$, Eq. (58) reduces to the well-known Randall–Sundrum form (we ignore $\kappa/a^2, \Lambda_{\rm eff}$ and C/a^4 which are unimportant for our purposes)

$$H^2 = \frac{8\pi G_{\rm N}}{3}\rho\left(1 + \frac{\rho}{2\sigma}\right) \,. \quad (59)$$

On the other hand if $\epsilon = -1$ then we obtain

$$H^2 = \frac{8\pi G_{\rm N}}{3}\rho\left(1 - \frac{\rho}{2|\sigma|}\right) \,, \quad (60)$$

where the brane tension $\sigma < 0$. (The two braneworld models (59) and (60) are dual, as demonstrated in [99].) Equation (60) clearly demonstrates that $H \simeq 0$ when $\rho_{\rm bounce} = 2|\sigma|$ *i.e.* the universe *bounces* when the energy density of matter becomes sufficiently large. It is important to note that the singularity-free nature of the early universe is a generic outcome of this theory and does not depend upon whether or not matter violates the energy conditions. An example of a bouncing braneworld universe is shown in figure (7) below.

An early time bounce, such as (60), can be used to construct cyclic models of the universe [100, 101, 102, 103]. Consider for instance the mechanism proposed in [100] in which Phantom dark energy is presumed to exist in addition to matter and radiation so that

$$\rho = \rho_r + \rho_m + \rho_P \quad (61)$$

where $\rho_r \propto a^{-4}$, $\rho_m \propto a^{-3}$, $\rho_P \propto a^{-3(1+w)}$ and $w < -1$ in the case of Phantom leading to

$$\rho = \frac{A}{a^4} + \frac{B}{a^3} + Ca^{3(|1+w|)} \,. \quad (62)$$

It is clear from (62) that ρ grows at small *as well as* large values of $a(t)$. Indeed within one cycle the value of H will pass through zero twice: (i) at early times when the bounce in (60) is brought about due to large values of

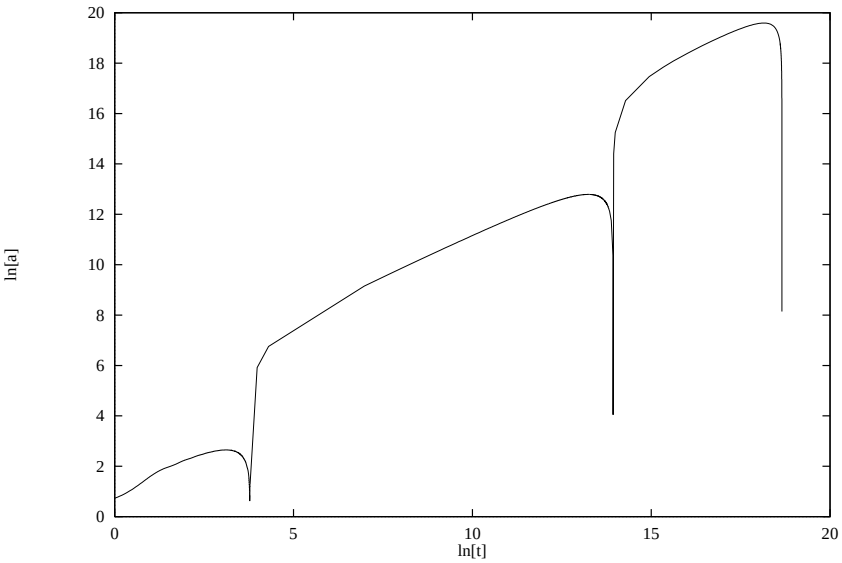

Fig. 7. A non-singular cyclic universe with ever-increasing successive expansion maxima. Figure courtesy of [101].

the radiation density and (ii) during late times, when the large value of the Phantom energy leads to $H = 0$ in (60) and initiates the universe's recollapse.

Similarly, it is easy to show that a bouncing spatially closed universe with matter satisfying $\rho + 3p > 0$ will also be 'cyclic' in the sense that it will pass through an infinite number of nonsingular expanding-collapsing epochs [101, 102]. As demonstrated in [101, 102], a massive scalar field in such a universe usually leads to an increase in the amplitude of consecutive expansion cycles and to a gradual amelioration of the flatness problem; see figure 7.

4 Model-independent reconstruction of Dark Energy

In view of the very large number of dark energy models, some of which have been discussed in this review, it becomes meaningful to ask whether we can reconstruct the properties of DE from observations in a model independent manner. This question is especially important since, as we have seen, dark energy models can be constructed by changing both the matter sector as well as the gravity sector of the theory. To the former category belong quintessence, Chaplygin gas etc. and to the latter braneworld models, k-essence, scalar tensor gravity, f+ f(R) theories etc.

We begin our discussion of cosmological reconstruction of the properties of dark energy by noting that the the Hubble parameter is related to the

luminosity distance [92, 104]

$$H(z) = \left[\frac{d}{dz}\left(\frac{d_L(z)}{1+z}\right)\right]^{-1}, \tag{63}$$

and that the scalar field potential (in quintessence) as well as its equation of state can be directly expressed in terms of the Hubble parameter and its derivative [92, 104]

$$\frac{8\pi G}{3H_0^2}V(x) = \frac{H^2}{H_0^2} - \frac{x}{6H_0^2}\frac{dH^2}{dx} - \frac{1}{2}\Omega_m\, x^3, \tag{64}$$

$$\frac{8\pi G}{3H_0^2}\left(\frac{d\phi}{dx}\right)^2 = \frac{2}{3H_0^2 x}\frac{d\ln H}{dx} - \frac{\Omega_m x}{H^2},\ x = 1+z, \tag{65}$$

$$\tag{66}$$

$$w_\phi(x) \equiv \frac{p}{\varepsilon} = \frac{(2x/3)d\ln H/dx - 1}{1 - (H_0^2/H^2)\,\Omega_m x^3}. \tag{67}$$

Both the quintessence potential $V(\phi)$ as well as the equation of state $w_\phi(z)$ may therefore be reconstructed provided the luminosity distance $d_L(z)$ is known to reasonable accuracy from observations.

In practice it is useful to have an ansatz for one of the three quantities: $d_L(z)$, $H(z)$ or $w(z)$, which can then be used for cosmological reconstruction. Popular fitting functions encountered in the literature include:
An ansatz for the dark energy [105]

$$\rho_{\rm DE}(x) = \sum_{i=0}^{N} A_i x^i, \quad x = 1+z\ . \tag{68}$$

Fitting functions to the dark energy equation of state [106, 107]:

$$w(z) = \sum_{i=0}^{N} w_i z^i\ ,$$
$$w(z) = w_0 + \frac{w_1 z}{1+z}\ . \tag{69}$$

The fitting parameters w_i, A_i are obtained by matching to observations. In principle only the first few terms in either series (68), (69) is used since the current Sn data is quite noisy; see [106, 108, 109] for a discussion of these issues. Other fitting functions have been discussed in [112, 106, 113, 114, 108] and a summary of different approaches to cosmological reconstruction may be found in [109].

An example of the cosmological reconstruction of the equation of state of dark energy from Sn data is shown in figure (8) for the ansatz (68); see also [110, 111]. From this figure we see that the combined Supernova+CMB data

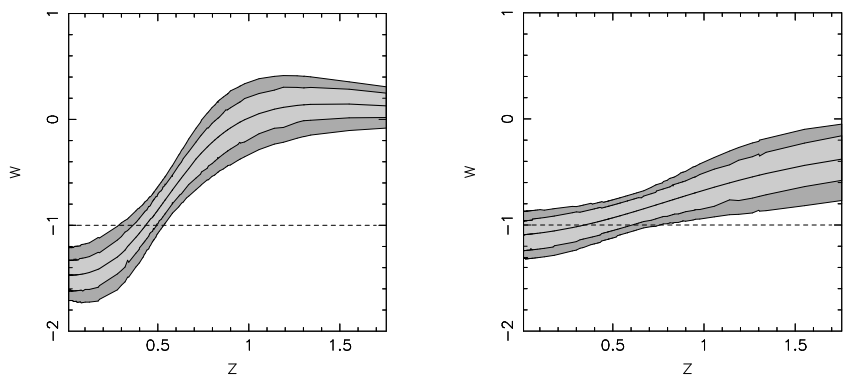

Fig. 8. A model independent analysis of SNe(Gold)+CMB data, using ansatz (68). Ω_m and h are fixed at best-fit values of $\Omega_m = 0.385, h = 0.60$ (left panel) and WMAP priors of $\Omega_m = 0.27 \pm 0.04$ and $h = 0.71 \pm 0.06$ (right panel). The joint CMB+Sn analysis therefore appears to favour an evolving equation of state for dark energy if the Hubble parameter is low and the matter density high (left panel). Strong priors on h and Ω_m on the other hand imply less evolution for dark energy and give results which are in better agreement with LCDM (shown by the dashed horizontal line). From Alam, Sahni and Starobinsky [116].

favour evolving dark energy for higher values of $\Omega_m \simeq 0.38$, and lower values of $h \simeq 0.6$. For $\Omega_m = 0.3$ and $h = 0.7$ the cosmological constant is in very good agreement with observations.

It should also be noted that the equation of state is not a fundamental physical quantity for dark energy models based either on non-Einsteinian gravity (scalar-tensor theories, Cardassian approximation, k-essence, R + f(R) gravity, etc.) or on Einstein gravity in more than 3+1 dimensions, or indeed in cosmological models with explicit interaction terms between dark matter and dark energy (see [109, 2] for discussions of this issue). In all these cases it is useful to supplement the effective equation of state by geometrical variables, which are 'model independent', in the sense that they do not depend upon an underlying theory of gravity in an explicit way, but can be constructed solely from the expansion factor and its derivatives, and therefore would be expected to apply to all metric theories of gravity defined on a space-time manifold. In all such theories the universe can be expected to be characterised by the following general form for the expansion factor:

$$a(t) = a(t_0) + \dot{a}\big|_0 (t - t_0) + \frac{\ddot{a}\big|_0}{2}(t - t_0)^2 + \frac{\dddot{a}\big|_0}{6}(t - t_0)^3 + \ldots . \qquad (70)$$

Usually dark energy models such as quiessence, quintessence, k-essence, braneworld models, Chaplygin gas etc. give rise to families of curves $a(t)$ hav-

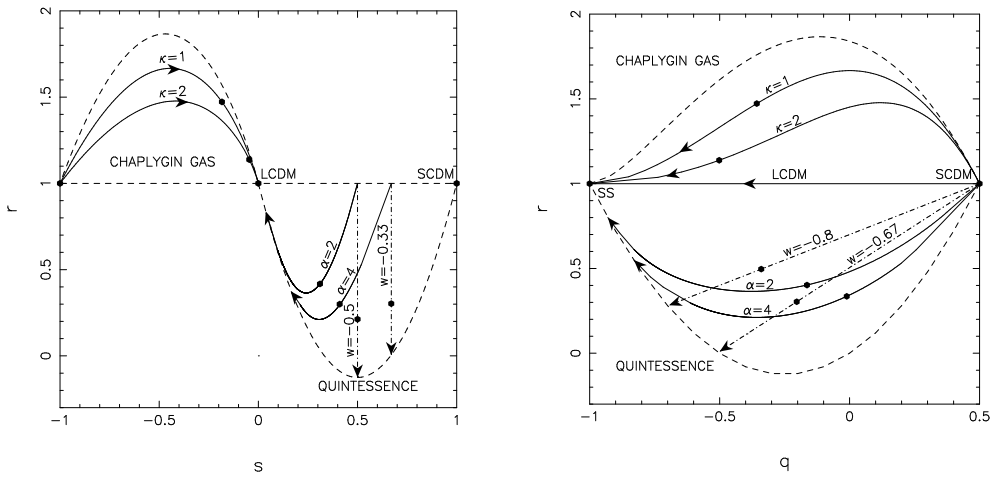

Fig. 9. The left panel (a) shows the time evolution of the statefinder pair $\{r, s\}$ for quintessence models and the Chaplygin gas. Quintessence models lie to the right of the LCDM fixed point ($r = 1, s = 0$) (solid lines represent tracker potentials $V = V_0/\phi^\alpha$, dot-dashed lines representing quiessence with constant equation of state w). For quiessence models, s remains constant at $1+w$ while r declines asymptotically to $1+\frac{9}{2}w(1+w)$. For tracker models, s monotonically decreases to zero whereas r first decreases from unity to a minimum value, then rises to unity. These models tend to approach the LCDM fixed point ($r = 1, s = 0$) from the right at $t \to \infty$. Chaplygin gas models (solid lines) lie to the left of the LCDM fixed point. The right panel (b) shows the time evolution of the pair $\{r, q\}$, where q is the deceleration parameter. The solid line, which corresponds to the time evolution of the LCDM model, divides the $r-q$ plane into two halves. The upper half is occupied by Chaplygin gas models, while the lower half contains quintessence models. All models diverge at the same point in the past ($r = 1, q = 0.5$) which corresponds to a matter dominated universe (SCDM), and converge to the same point in the future ($r = 1, q = -1$) which corresponds to the steady state model (SS) – the de Sitter expansion (LCDM $\to$ SS as $t \to \infty$ and $\Omega \to 0$). The dark dots on the curves show current values $\{r_0, s_0\}$ (left) and $\{r_0, q_0\}$ (right) for different dark energy models. In all models, $\Omega = 0.3$ at the current epoch. In both panels quiessence is shown as dot-dashed while dashed lines mark envelopes for Chaplygin gas (upper) and quintessence (lower). Figure courtesy of [109].

ing quite different properties. Since we know that the acceleration of the universe is a fairly recent phenomenon we can, in principle, confine our attention to small values of $|t - t_0|$ in (70).

Accordingly, as dicussed in [105, 109], a new diagnostic of dark energy called 'statefinder' can be constructed using both the second and third derivatives of the expansion factor (see also [117, 118, 119]). It should be noted that the second derivative is encoded in the deceleration parameter which has the

following form in a spatially flat universe:

$$q = -\frac{\ddot{a}}{aH^2} \equiv \frac{1}{2}\left(1 + 3w\Omega_X\right) \ , \quad \Omega_X = 1 - \Omega_m \ . \tag{71}$$

The statefinder pair $\{r, s\}$, defines two new cosmological parameters (in addition to H and q):

$$r \equiv \frac{\dddot{a}}{aH^3} = 1 + \frac{9w}{2}\Omega_X(1+w) - \frac{3}{2}\Omega_X\frac{\dot{w}}{H} \ , \tag{72}$$

$$s \equiv \frac{r-1}{3(q-1/2)} = 1 + w - \frac{1}{3}\frac{\dot{w}}{wH} \ . \tag{73}$$

In figure 9 we show the statefinder pair for several dark energy models including LCDM, quintessence and the Chaplygin gas.

Clearly the Statefinders succeed very well in discriminating between rival models of dark energy and should therefore prove to be a very potent diagnostic of the expansion of the universe especially after bigger supernova data sets (such as SNAP) become available.

5 Conclusions

The nature of dark energy is clearly one of the key enigma's to confront physicists during the past decade. It could well be that the nature of dark energy turns out to be somehow related to dark matter (which, like dark energy, is non-baryonic) or perhaps even to the inflaton field, since the latter – like dark energy – also disobeys the strong energy condition. It is indeed a mystery as to why the Universe accelerated during *very early times* and then again, much later, during the present epoch. It is commonly believed that the equation of state of the inflaton which drives inflation varies with time. Indeed, this is essential if the early accelerating phase is to end and be succeeded by the radiative epoch. The situation concerning dark energy is not quite as clear, and although several compelling dynamical models of dark energy have been advanced it remains true that the cosmological constant provides very good agreement to current observational data. Indeed, it might be fair to say that, from the theoretical standpoint, the single most important question to be asked of dark energy is

$$\text{Is } w = -1 \ ?$$

Answering this question in the affirmative would imply that, in all likelyhood, the cosmological constant is the vacuum energy. One would then need to review the cosmological constant problem again, in order to fathom why the formally infinite quantity $\langle T_{ik} \rangle$ is in fact so very small, as implied by observations.

On the other hand, if either $w \neq -1$ or if the DE density is shown to be time dependent, then this would throw up fresh avenues for DE research. In

all likelyhood the cosmological constant problem would need to be decoupled from the DE conundrum and searches for evolving DE models which produce $\rho_{\rm DE} \simeq 10^{-47}{\rm GeV}^4$ without exacerbating 'cosmic coincidence' would need to be re-examined. In either case the key to determining the properties of DE to great precision clearly lies with ongoing and future astrophysical experiments and observations.

In this connection it is heartening to know that, since the original discovery of an accelerating universe [125, 126, 127] the Sn data base has grown considerably and data pertaining to ~ 200 type Ia supernovae is currently avaliable in the literature [129, 58, 130, 128]. It is however of paramount importance that Sn observations continue to be supplemented by other investigations which are sensitive to the geometry of space and can be used as independent tests of the DE hypothesis. The CMB, volume-redshift test, Sunyaev-Zeldovich surveys, Alcock-Paczynski test, the angular size-redshift test and gravitational lensing have all been suggested as possible probes of dark energy, and will doubtless enrich the theory vs observations debate in the near future. In addition, the proposed SNAP satellite which aims to measure light curves of ~ 2000 supernovae within a single year [131], should provide a big step forward in our understanding of type Ia supernovae and help determine the cosmological parameters to great precision.

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