COSMOLOGY

IUCAA-NCRA Graduate School
(January - February 2000)

Assignment 1
Due in class Friday, January 28, 2000

1. *Achieving the Impossible:* [10 marks]
Show that the line element

$$ds^2 = dt^2 - e^{2Ht}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right] \tag{1}$$

representing a FRW universe with $k = 0, a(t) = e^{Ht}$ can also be expressed in the form

$$ds^2 = dT^2 - \cos h^2 HT\left[\frac{dR^2}{1-R^2} + R^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] \tag{2}$$

which has $a = \cosh HT$ but $k = 1$!! But haven't you heard people talk of $k = 1$ as "closed universe" and $k = 0$ as "open universe" with some invariant significance? Is this universe closed or open?

2. *Conformal flatness of FRW universes*: [20 marks]
Find a transformation which will express a general FRW universe (with $k \neq 0$)in a conformally flat form.

3. *Redshift in a perturbed FRW universe:* [25 marks]
(a) Suppose the metric of the spacetime is slightly different from that of a Friedmann model and is given by

$$ds^2 = dt^2 - a^2(t)\left[\delta_{\alpha\beta} - h_{\alpha\beta}(t,x)\right]dx^\alpha dx^\beta. \tag{3}$$

where $h_{\alpha\beta}$ is a small perturbation. A photon emitted by a distant source propagates through this spacetime and is received by an observer at the present epoch. Show that, to the lowest order in $h_{\alpha\beta}$, the quantity ωa [where ω is the frequency of the photon] will vary by an amount

$$\ln\left(\frac{\omega_2 a_2}{\omega_1 a_1}\right) = \frac{1}{2}\int_{\text{along path}} \dot{h}_{\alpha\beta} n^\alpha n^\beta dt \tag{4}$$

where the trajectory of the photon is

$$x^\alpha = n^\alpha \int_{t_1}^{t_2} \frac{dt'}{a(t')}. \tag{5}$$

and the integral in (4) is taken along the trajectory of the photon.

(b) A bullet is shot out into an expanding Robertson-Walker universe with a velocity V_1 (relative to cosmological observers). Later, when the universe has expanded by a scale factor $(1+z)^{-1}$, it has a different velocity V_2 with respect to cosmological observers. Find V_2 in terms of z and V_1. Show that in the limit $V_1 \to c$, the formula for the redshift of photons is obtained.

(c) A bullet is shot into an expanding Friedmann universe. When $k = -1$, show that it approaches the same velocity as some cosmological observer, but a position which is a constant proper distance away from him. When $k = 0$, show that the bullet again approaches the velocity of some cosmological observer but that the proper distance between the bullet and that observer becomes arbitrarily large as $t \to \infty$.

4. *Counting Things:* [10 marks]

(a) Suppose the number of galaxies per unit proper volume within the redshift range $(z, z+dz)$ is $n(z)$. [If the galaxies are neither created or destroyed, $n(z) = n(0)(1+z)^3$.] Let the effective cross section of a galaxy be πr_G^2. Find an expression for the probability that a galaxy with a redshift in the range $(z, z+dz)$ intervenes in our line of sight to a distant source of redshift z_s. Hence calculate the optical depth due to galaxies for a source located at a redshift of z_s.

Observations suggest that the effective radius of a galaxy is $r_G \approx 10h^{-1}$ kpc and the comoving number density at $z = 0$ is $n_G \simeq 0.02h^3\text{Mpc}^{-3}$. Estimate the fraction of the sky which is covered by galaxies at $z = 1$ in a matter dominated universe. (Take $\Omega = 1$ and assume that galaxies are conserved.) At what redshift does the optical depth due to galaxies become unity?

(b) Calculate the expected number density of galaxies per unit redshift interval, per unit solid angle $(dN_G/dzd\Omega)$ along any line of sight in a matter dominated universe.

(c) If an idealized $k = 0$, matter dominated, Friedmann universe with Hubble constant H_0 contains homogeneously distributed sources of constant luminosity L, and if the local number density of such sources in space is now n, what is the brightness B of the night sky (energy per steradian of sky per collection area per time)? [If the universe were static and infinitely old the brightness would be infinite; this is called Olber's Paradox.]

5. *Cosmic Time-Redshift Relation:* [15 marks]

Newspaper articles often report cosmological discoveries with words similar to these: "Astronomers have discovered a startling galaxy that should not exist according to present theories. In only one billion years after the beginning of the universe, this cannibal has already grown larger than the Milky Way." Suppose that you are asked by an astronomer friend what is the redshift of galaxy. Estimate the redshift at $t = 1Gyr$ for the following three cosmological models. In each case, assume that the Hubble constant is $70kms^{-1}Mpc^{-1}$.

a) SCDM: flat $\Omega_{cdm} = 1$; (Standard cold dark matter)

b) OCDM: open $\Omega_{\rm cdm} = 0.3$; (Open cold dark matter)

c) $\wedge$CDM: flat $\Omega_{\rm cdm} = 0.35, \Omega_{\wedge} = 0.65$; (lambda + cold dark matter)

6. *Thermal history:* [20 marks]

(a) Assume that we live in a Robertson-Walker universe with matter, radiation and curvature. The present mass density is $\rho_m = 3\Omega_0 H_0^2/(8\pi G)$, where $H_0 = 100hkms^{-1}Mpc^{-1}$ and $\Omega_0 \leq 1(i.e., \mathrm{k} \leq 0)$. The present radiation temperature is $T_0 = 2.73K$. Assume that only photons, with present temperature $T_0 = 2.73K$ contribute to the radiation; ignore neutrinos. At what redshift did the radiation energy density equal the matter density? How old was the universe then and what was the radiation temperature? Your answers should scale appropriately with Ω_m and h.

(b) In terms of the conserved (baryon / photon) ratio η find the temperature at which hydrogen would have recombined, (i.e. had an equilibrium ionization fraction of 0.5) assuming thermodynamic equilibrium. Evaluate your answer for $\eta = 3 \times 10^{-10}$, the value favoured by primordial nucleosynthesis. What is the corresponding redshift of recombination, assuming $T_0 = 2.73K$? How does it change if $\eta = 1 \times 10^{-10}$?

(c) Under which of the following suppositions would primordial nucleosynthesis have produced less ^{4}He than the standard model? Less ^{2}H (deuterium) ?

(i) Suppose that the baryon density in the universe today is larger than we think.

(ii) Suppose that there is a fourth neutrino flavour with mass much less than 1 MeV. (It must be "sterile", e.g. right-handed, not to have been detected in Z^0 decay at LEP.)

(iii) Suppose that there are many more neutrinos than antineutrinos or photons in the cosmic background today.

(iv) Suppose that there are many more antineutrinos than neutrinos or photons in the cosmic background today.

(v) Suppose that there is a significant contribution of gravitational radiation to the total energy density of the universe, comparable in magnitude with the energy density of the microwave background radiation.