COSMOLOGY

IUCAA-NCRA Graduate School
(January - February 2000)

Assignment 2
Due in class , February 11, 2000

1. *Angle subtended by causal horizon:* [25 marks]

For this problem, use the conformal time coordinate τ defined by $dt = a d\tau$ and the 'angular' variable χ defined by $d\chi = (1 - kr^2)^{-1/2}$ dr. Consider two comoving observers A and B, each at the same comoving distance χ from you, the observer. At conformal time τ_e, A and B come into causal contact for the first time: a photon emitted by A at $\tau = 0$ reaches B (and vice versa) at $\tau = \tau_e$. At $\tau = \tau_e$ both A and B emit photons along radial paths toward you which you receive today (at $\tau = \tau_0$). For simplicity, ignore the gravitational effect of radiation in the universe; this is a fair approximation for $z \ll 10^4$.

(a) Assuming that you live in an $\Omega = 1$ matter dominated universe, what angular separation do you measure between A and B? (you may use small-angle formulae.)

(b) Evaluate the angle in degrees if $1 + z_e = 1300$ so that A and B are on the cosmic photosphere at recombination (when the cosmic plasma became transparent).

(c) Redo part (b) for the OCDM and ΛCDM models with $\Omega_m = 0.35 = \Omega_{\rm tot}$ and $\Omega_m = 0.35, \Omega_{\rm vac} = 0.65, \Omega_{\rm tot} = 1$. For the same Ω_m in both models, does the angle subtended by the causal horizon at recombination depend more strongly on spatial curvature or $\Omega_{\rm vac}$?

2. *Thomson optical depth* : [15 marks]

(a) Derive the contribution to the Thomson optical depth $(d\tau_T/dlna)$ per logarithmic interval of expansion factor, assuming that the universe is matter-dominated at the time of interest but that $\Omega_m \neq 1$. Assume that helium is neutral but hydrogen is partially ionized with ionization fraction x_e. Your answer should depend on $a, x_e, \Omega_m, \Omega_B, h$, and Y.

(b) For a Friedmann model (mattter plus curvature only), integrate the result of part (a) to get the total optical depth to redshift z in the limit $x_e = 1$ for all z (no recombination). For an OCDM model with $\Omega_B h^2 = 0.019, \Omega_m = 0.3, \Omega_k = 0.6$, and $h = 0.65$, at what redshift would the Thomson optical depth equal unity if recombination never occurred?

3. *Life and times at* $z = 1000$ *:* [25 marks]

The temperature, $T_{\rm atom}$, at which hydrogen atoms are formed can be computed if we make two simplifying assumptions: (i) The system is in thermodynamic equilibrium and (ii) the recombination proceeds through the electron and proton combining to form a hydrogen atom in the ground state. (Neither of these assumptions is quite correct; we discussed the modifications which are required in a more realistic scenario in the class).

(a) Under these assumptions, show that

$$(1 + z_{\rm rec}) \cong 1367[1 - 0.024 \ln(\Omega_B h^2)]^{-1}. \tag{1}$$

For $\Omega_B h^2 = 1, 0.1, 0.01$ we get $T_{\rm rec} \cong 0.324$eV, 0.307eV, 0.292eV respectively. These values correspond to the redshifts of $1367, 1296$ and 1232. In a "hot" Friedmann cosmology, two unrelated important epochs are when matter first begins to dominate radiation ($\rho_{\rm matter} \approx \rho_{\rm radiation}$), and when protons and elctrons recombine to form hydrogen. Given that these epochs happen to be nearly the same in our universe, deduce a numerical value for the conserved entropy per baryon $\sigma \equiv 4a_s T^3/3n$ (where T = temperature, a_s = radiation constant, n = number density of baryons).

(b) For the above process to be described by equilibrium thermodynamics, the reaction rate $p + e \to H + \gamma$ should be higher than the expansion rate. Estimate the temperature T_D below which this condition fails, if $\Omega \approx 1$ and $\Omega_B \approx 0.1$. What fraction of electrons (and protons) will remain as free charged particles at $T \ll T_D$?

(c) When most of the electrons and protons combine to form neutral atoms, the mean-free-path of photons (due to Thompson scattering) will become larger than $d_H(t)$. Estimate the temperature $T_{\rm dec}$ at which this happens.

(d) After decoupling, the free electrons will continue to exchange energy with radiation. Collisions between e, p and H will distribute this energy among the constituents of the (nearly neutral) plasma. Estimate the redshift z_c upto which this process can keep the temperature of matter and radiation equal. (Assume that $\Omega \approx 1$ and $\Omega_B h^2 \approx 0.015$).

4. *Statistical indicators*: [10 marks]

Express (i)$\sigma_{STH}(R)$ as an integral over $\xi(x)$; (ii)$\sigma_{STH}(R)$ as an integral over $\bar{\xi}(x)$; (iii)P(k) as an integral over $\bar{\xi}(x)$ and (iv) vice-versa.

5. *Mezzaros Effect*: [25 marks]

Perturbations which are of interest in cosmology have comoving wavelengths less than $\lambda_{\rm eq} \simeq 13(\Omega h^2)^{-1}$ Mpc. They enter the Hubble radius in the radiation dominated phase.

These perturbations were growing as a^2 before entering the Hubble radius. When $\lambda \ll d_H$, we can use Newtonian equations to study their growth. In this case, the equations are :

$$\ddot{\delta}_{\mathbf{k}} + \frac{2\dot{a}}{a}\dot{\delta}_{\mathbf{k}} \cong 4\pi G \rho_{\rm DM} \delta_{\mathbf{k}} \tag{2}$$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\left(\rho_R + \rho_{\rm DM}\right) \tag{3}$$

where $\rho_{\rm DM}$ denotes the background dark matter density and ρ_R is the background radiation density.

(a) Solve this equation and show that the two independent solutions are

$$\begin{aligned} \delta_g &= 1 + \frac{3}{2}x \\ \delta_d &= \left(1 + \frac{3x}{2}\right) \ln\left[\frac{(1+x)^{1/2}+1}{(1+x)^{1/2}-1}\right] - 3(1+x)^{1/2}. \end{aligned} \tag{4}$$

Here $x = a/a_{\rm eq}$ and we have suppressed the subscript $\mathbf{k}$ in $\delta_{\mathbf{k}}$ for simplicity of notation. How does δ behave for small x? Large x? Why?

(b) Combining with the earlier result we can determine the evolution of density perturbations in dark matter during all relevent epochs. Assume that the dark matter perturbation was given by $\delta(x) = x^2$ for $x \ll 1$. Use appropriate matching conditions when the mode enters the Hubble radius and determine the total growth for the perturbation from the time it enters the Hubble radius *until* the universe becomes matter dominated. Assume that $a_{\rm enter} \ll a_{\rm eq}$ and show that

$$\frac{\delta_{\rm final}}{\delta_{\rm enter}} \equiv \frac{\delta(a_{\rm eq})}{\delta(a_{\rm enter})} \simeq 5\ln\left(\frac{\lambda_{\rm eq}}{\lambda}\right) \tag{5}$$

(c) In the discussion above, we have assumed that $\Omega = 1$, which is a valid assumption in the early phases of the universe. During the later stages of evolution in a matter dominated phase, we have to take into account the actual value of Ω. Solve equation (2) for a matter dominated universe with arbitrary Ω and demonstrate that the decaying solution $\delta_d(t)$ and the growing solution $\delta_g(t)$ are given by

$$\delta_d(t) = H(t); \qquad \delta_g(t) = H(t)\int \frac{dt}{a^2 H^2(t)} \equiv b(t) \tag{6}$$

where $H(t) = (\dot{a}/a)$.

(d) If $\mathbf{v} = a\dot{\mathbf{x}}$ is the peculiar velocity of the fluid and $\mathbf{g}$ is the acceleration produced by the perturbed matter, show that

$$\mathbf{v} = \frac{2f}{3H\Omega}\mathbf{g} \tag{7}$$

where $f = (a/\delta)\,(d\delta/da)$, $H(t) = (\dot{a}/a)$ and $\Omega(t) = (\rho/\rho_c)$.
(e) Calculate δ_g explicitly for a matter dominated universe with arbitrary Ω and show that it can be expressed as

$$\delta_g(z) = \frac{1 + 2\Omega + 3\Omega z}{(1-\Omega)^2} - \frac{3}{2}\frac{\Omega\,(1+z)\,(1+\Omega z)^{1/2}}{(1-\Omega)^{5/2}} \ln Q(z) \tag{8}$$

where

$$Q(z) = \frac{(1+\Omega z)^{1/2} + (1-\Omega)^{1/2}}{(1+\Omega z)^{1/2} - (1-\Omega)^{1/2}}. \tag{9}$$

The redshift z is related to the expansion factor $a(z) = a_0(1+z)^{-1}$. How does this solution behave for large and small z?