# COSMOLOGY

IUCAA-NCRA Graduate School
(January - February 2000)

**Assignment 3**
**Due on March 1, 2000**

**1.** *Density contrast from the trajectories:* [25 marks]
One is often interested in the evolution of the density contrast $\delta\left(t, \mathbf{x}\right)$ rather than on the trajectories. Since the density contrast can be expressed in terms of the trajectories of the particles, it should be possible to write down a differential equation for $\delta(t, \mathbf{x})$ based on the equations for the trajectories $\mathbf{x}_i(t)$ derived in the class. It is, however, somewhat easier to write down an equation for $\delta_{\mathbf{k}}(t)$ which is the fourier transform of $\delta(t, \mathbf{x})$. Show that $\delta_{\mathbf{k}}(t)$ can be expressed in terms of the positions of the individual particles via the relation:

$$\delta_{\mathbf{k}}(t) = \frac{1}{N}\sum_i \exp\left[i\mathbf{k}\cdot\mathbf{x}_i(t)\right] \tag{1}$$

where $N$ is the total number of particles in a large volume $V$ under consideration. Using the equations of motion for $\mathbf{x}_i(t)$ show that $\delta_{\mathbf{k}}$ satisfies the equation

$$\ddot{\delta}_{\mathbf{k}} + 2\frac{\dot{a}}{a}\dot{\delta}_{\mathbf{k}} = 4\pi G\rho_b\delta_{\mathbf{k}} + A - B \tag{2}$$

with

$$\begin{aligned} A &= 2\pi G\rho_b \sum_{\mathbf{k}'\neq 0,\mathbf{k}} \delta_{\mathbf{k}'}\delta_{\mathbf{k}-\mathbf{k}'}\left\{\frac{\mathbf{k}\cdot\mathbf{k}'}{k'^2} + \frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^2}\right\} \\ B &= \frac{1}{N}\sum_j \left(\mathbf{k}\cdot\dot{\mathbf{x}}_j\right)^2 \exp\left[i\mathbf{k}\cdot\mathbf{x}_j(t)\right]. \end{aligned} \tag{3}$$

Here overdot represents derivative with respect to $t$.

**2.** *The 'pressureless dust'* : [20 marks]
The crudest form of aproximation to study the evolution of density inhomogeneities in expanding universe is obtained by ignoring the velocity dispersion.
(a) Show that the equations describing such a "pressureless dust" are

$$\frac{\partial\delta}{\partial b} + \partial_i\left[u^i(1+\delta)\right] = 0; \quad \nabla^2\psi = \left(\frac{\delta}{b}\right) \tag{4}$$

$$\frac{\partial u^i}{\partial b} + u^k \partial_k u^i = -\frac{3A}{2b}\left[\partial^i \psi + u^i\right]; \quad A = \left(\frac{\rho_b}{\rho_c}\right)\left(\frac{\dot{a}b}{a\dot{b}}\right)^2. \tag{5}$$

Here $u^i = u^i(b, \mathbf{x})$ is the velocity of the fluid at "time" $b$ and location $\mathbf{x}$; since the velocity dispersion has been ignored, $u^i$ satisfies the same equation as the velocity $w^i$ of the individual particles. All the other vaiables were defined in the class.

(b) Consider the evolution of density contrast in the fluid limit. Decompose the derivative of the velocity $\partial_a u_b$ into shear $\sigma_{ab}$, rotation $\Omega^c$ and expansion $\theta$ by writing

$$\partial_a u_b = \sigma_{ab} + \epsilon_{abc}\Omega^c + \frac{1}{3}\delta_{ab}\theta. \tag{6}$$

Explain why such a decomposition is natural. Show that the fluid equations can be combined to give

$$\frac{d^2\delta}{db^2} + \frac{3A}{2b}\frac{d\delta}{db} - \frac{3A}{2b^2}\delta(1+\delta) = \frac{4}{3}\frac{1}{(1+\delta)}\left(\frac{d\delta}{db}\right)^2 + (1+\delta)(\sigma^2 - 2\Omega^2) \tag{7}$$

$\sigma^2 = \sigma_{ab}\sigma^{ab}$ and $\Omega^2 = \Omega_a\Omega^a$.

**3.** *Spherical Top Hat:* [40 marks]

In the nonlinear regime when $\delta \gtrsim 1$ it is not possible to solve equations for pressureless dust exactly. Some progress, however, can be made if we assume that the solutions are spherically symmetric. We consider, at some initial epoch $t_i$, a spherical region of the universe which has a slight constant overdensity compared to the background. As the universe expands, the overdense region will expand more slowly compared to the background, will reach a maximum radius, contract and virialize to form a bound nonlinear system. Such a model is called "spherical top-hat".

(a) Show that, in the case of spherical symmetry, the density contrast $\delta$ satisfies the equation

$$\ddot{\delta} - \frac{4}{3}\frac{\dot{\delta}^2}{(1+\delta)} + \frac{2\dot{a}}{a}\dot{\delta} = 4\pi G\rho_b\delta\,(1+\delta) \tag{8}$$

where dot denotes $(d/dt)$. This is easily done from the results of last problem but I would like you to derive it from the results of Problem 1 using suitable trajectories.

(b) Further show that the solution to (8) can be expressed in the form

$$\delta(t) = \frac{2GM}{\Omega_m H_0^2 a_0^3}\left[\frac{a(t)}{R(t)}\right]^3 - 1 \equiv \lambda\left(\frac{a}{R}\right)^3 - 1 \tag{9}$$

where $M$ is a constant and $R(t)$ satisfies the equation

$$\ddot{R} = -\frac{GM}{R^2} - \frac{4\pi G}{3}\left(\rho + 3p\right)_{\text{rest}} R \tag{10}$$

in which $(\rho + 3p)_{\text{rest}}$ is due to components other than dust like matter. Interpret this equation.

(c) Assume that the background is described by a $\Omega = 1$, matter dominated, universe for which $b = a = (t/t_0)^{2/3}$. In this case, show the evolution of density contrast can be expressed in the form $1 + \delta(a) = \left(2GM/H_0^2\right)\left[a^3/R^3(a)\right]$ where $R(t)$ satisfies the equation

$$\frac{d^2R}{dt^2} = -\frac{GM}{R^2} \tag{11}$$

and $M$ is a constant. Let $R(t_i) = r_i$ where $t_i$ is the time at which initial conditions are specified. Let $M = (4\pi/3)\, r_i^3 \rho_b(t_i)(1 + \delta_i)$, with $\delta_i > 0$. Further assume that $\left(\dot{R}^2/2\right)_{t=t_i} = (1/2)\, H_i^2 r_i^2$.

(i) Show that the evolution of the density contrast $\delta(z)$ is given implicitly by the functions $[\delta(\theta), z(\theta)]$ with

$$(1 + z) = \left(\frac{4}{3}\right)^{2/3} \frac{\delta_i (1 + z_i)}{(\theta - \sin\theta)^{2/3}} = \left(\frac{5}{3}\right)\left(\frac{4}{3}\right)^{2/3} \frac{\delta_0}{(\theta - \sin\theta)^{2/3}}; \tag{12}$$

$$\delta = \frac{9}{2}\frac{(\theta - \sin\theta)^2}{(1 - \cos\theta)^3} - 1 \tag{13}$$

where $\delta_i$ is the density contrast at the initial redshift $z_i$ and $\delta_0 = (3/5)\delta_i(1 + z_i)$ is the corresponding density contrast at the present epoch if the density perturbation grew according to linear theory. Also show that the density contrast at any epoch, predicted by the linear theory, is

$$\delta_L = \frac{3}{5}\left(\frac{3}{4}\right)^{2/3}(\theta - \sin\theta)^{2/3} \tag{14}$$

(ii) Compare the results for $\delta$ and $\delta_L$ at different epochs to estimate the accuracy of linear theory.

(iii) Prove that the spherical region reaches a maximum radius of $r_m = (3x/5\delta_0)$ at a redshift $z_m$ such that $(1 + z_m) \simeq (\delta_0/1.062)$ and the density contrast at maximum expansion is about 4.6.

(d) After reaching the maximum expansion, the matter will collapse inwards. It is likely that the system will get virialized during the collapse and form a gravitationally bound object. (i) Estimate the redshift $z_{\text{coll}}$ at which the virialization occurs and (ii) the density of the collapsed object.

**4.** *Spherical model with a density profile*: [15 marks]

(a) Explain qualitatively how the evolution described in the last problem gets modified if the initial density is not constant in the spherical region but is given by some profile $\rho_i(r)$ at $t = t_i$.

(b) Assume that the initial density profile is such that the energy $E(M)$ of a shell containing mass $M$ is given by a power law $E(M) = E_0(M/M_0)^{2/3-\epsilon}$. Describe the final density profile assuming that the evolution is self similar.