Problem Set 1

<u>To be returned before 11:30 am on Sept.20</u>

General Instructions

You have to return solutions to only the numbered problems. Starred problems are just for fun; you can do them if you like but you will not get any grades for that !

This paper has been set for 171 marks. The marks for each question (or subsection) are given in square brackets at the end of the question (or subsection). You will have to answer problems to get a maximum of 100. Any score obtained over 100 will *not* be carried over to the next assignment.

The solutions must contain some explanation as well and it will be nice if you outline your method of solving a given problem in the beginning of each solution.

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1. (i) Let the 4-velocity and the 4-acceleration vectors be defined as $u^i = dx^i/ds$ and $a^i = du^i/ds$ where x^i is the position 4-vector and s is the proper time. Express (a) d/ds in terms of d/dt, (b) u^0 in terms of v, (c) u^j ($j = 1, 2, 3$) in terms of $\mathbf{v}$ where $\mathbf{v}$ is the three velocity, (d) u^0 in terms of u^j, (e) v in terms of u^0 and (f) v^j in terms of u^j. [6]

(ii) Show that if two frames move with three velocities $\mathbf{v_1}$ and $\mathbf{v_2}$, their relative velocity is given by (Assume $c = 1$)

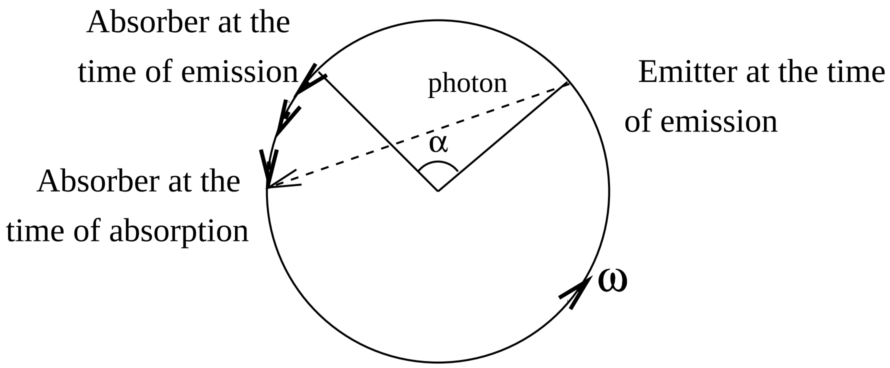
$$\mathbf{v}^2_{\mathbf{rel}} = \frac{(\mathbf{v_1} - \mathbf{v_2})^2 - (\mathbf{v_1} \times \mathbf{v_2})^2}{(1 - \mathbf{v_1} \cdot \mathbf{v_2})^2}$$

[9]

(iii) Define the "element of length in velocity space" dl_v as the relative velocity between two points corresponding to velocities $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$. From the formula given in part (ii) for the relative velocity, write down a formula for dl_v^2. Rewrite the above formula using spherical polar co-ordinates in the velocity space $(v,\, \theta,\, \phi)$ where v is the magnitude of the velocity and θ and ϕ are the polar angle and the azimuth of the direction of $\mathbf{v}$. If the substitution $v = \tanh\chi$ is made, how does the line element look in terms of $(\chi,\, \theta,\, \phi)$?

[2]

(iv) An emitter of photons and an absorber of photons are attached to the rim of a centrifuge of radius r at points separated by an angle α. The centrifuge rotates with an angular veocity ω. What is the redshift measured by the absorber as it detects the photons emitted by the emitter?



[6]

(v) Consider an observer moving along the x-axis with $a^i a_i = -g^2/c^4$ where a^i is the four acceleration. Interpret this motion as one of constant acceleration in a relativistically invariant way. Writing out the expression for $a^i a_i$ in the inertial frame in which the observer has the co-ordinates $(x,\, t)$, determine x as a function of t with initial conditions $x = v_x = 0$ at $t = 0$. What is the shape of the orbit in the $t - x$ plane?

Let the zero reading on a clock C_A carried by the accelerating observer and a clock C_I in the inertial frame coincide on earth at $t = 0$. What is the proper time shown by C_A when the accelerating observer moves $\sim 30,000$ light years to the center of the galaxy from the earth? What is the corresponding time shown by C_I? Assume that the proper acceleration of the observer is $10\, m/s^2$. [6]

2. Consider two Lorentz frames S and S' , with S' moving along the positive x-axis with velocity $\mathbf{V}$. You have seen in class how velocities transform. Write down the components of $\mathbf{v}$ as measured in the frame S in terms of the components of $\mathbf{v}'$ as measured in the frame S' of a particle. Choose the coordinate axes so that the velocity of the particle at any given moment lies in the XY plane. Write down the components of the velocity of the particle in the S and S' systems in terms of the magnitudes v and v' and the angles θ and θ' subtended with the X and X' axes respectively in the S and S' systems. Using the law of transformation of velocities write down the formula for $\tan\theta$ in the S frame in terms of $\sin\theta'$ and $\cos\theta'$ in the S' frame. This formula describes the change in the direction of the velocity on transforming from one reference system to another. [9]

3. Consider two Lorentz frames S and S' , with S' moving along the positive x-axis with velocity $\mathbf{v}$. We will call S' the rest frame and S the lab frame and denote quantities in these frames by the subscripts R and L. Assume that $v \lesssim c$ so that

$$\left(1 - \frac{v}{c}\right) \cong \frac{(1 - v^2/c^2)}{(1 + v/c)} \cong \frac{1}{2\gamma^2},$$

with $\gamma = (1 - v^2/c^2)^{-1/2} \gg 1$. Consider a photon with frequency ω_L travelling along the direction (θ_L, ϕ_L) in the lab frame.

(a) What is the direction (θ_R, ϕ_R) in which it will be seen as propagating in the rest frame? What is the frequency ω_R in the rest frame?

(b) Plot $\cos\theta_R$ against $\cos\theta_L$. Make a rough qualitative plot. Mark in the plot the direction of propagation in the two frames corresponding to (i) $\cos\theta_L = 1$; (ii) $\cos\theta_L = v/c$; (iii) $\cos\theta_L = 0$; (iv) $\cos\theta_L = -1$. For each of these cases relate ω_R to ω_L.

(c) What is the range of angles for which a photon propagating forward in one frame will appear to propagate forward in the other frame as well? What is the range of angles for which the photon is blueshifted? For what angle does the photon suffer no change in frequency?

(d) An object is emitting photons isotropically in its rest frame. Sketch qualitatively the angular dependence of the emission pattern as seen from the lab frame.

(e) A high energy particle is moving through an isotropic bath of radiation. Sketch the angular patern of collisions between the charge and photons in the rest frame of the particle. [14]

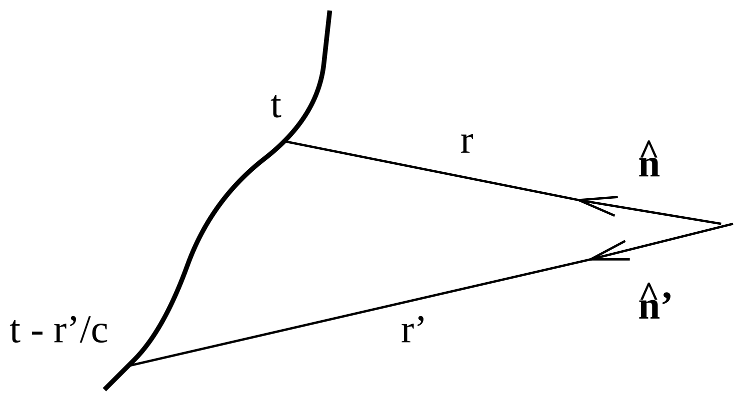
4. (a) Show that blackbody radiation appears as blackbody radiation in all inertial frames, though corresponding to different temperatures. Write down the transformation law for the temperature of a black body measured in this way.

(b) Use the formula just derived to predict the angular distribution of the temperature as seen by an object moving inside a large cavity with a uniform velocity.

(c) The temperature of the Cosmic Microwave Background Radiation (CMBR) follows this pattern. The observed angular distribution of the temperature of the CMBR is given by ($T \approx 2.73(1 + .0012\cos(\theta))$). Use this to find Earth's speed relative to the frame in which CMBR is isotropic (assume that the earth moves nonrelativistically). [6]

5. (a) The electric and magnetic fields of a moving charge are given by

$$\begin{aligned} \mathbf{E} &= -q\left[\frac{\hat{\mathbf{n}}'}{r'^2} + \frac{r'}{c}\frac{d}{dt}\left(\frac{\hat{\mathbf{n}}'}{r'^2}\right) + \frac{1}{c^2}\frac{d^2}{dt^2}\hat{\mathbf{n}}'\right] \\ c\mathbf{B} &= \hat{\mathbf{n}}' \times \mathbf{E} \end{aligned}$$



Find **E** for

(i) a charge at rest.

(ii) a charge moving with uniform velocity.

(b) The electric field $\mathbf{E}(\mathbf{x},\, t)$ and the magnetic field $\mathbf{B}(\mathbf{x},\, t)$ in a given region of space-time are functions of position and time. Consider the space-time event $\mathcal{P}$. By using test charges placed at $\mathcal{P}$, one can classically measure their velocities and accelerations and use the information to determine the electric and magnetic fields. What is the minimum number of test charges required to completely determine the fields? Justify your answer.

[14]

6. Express the acceleration $\dot{\mathbf{v}}$ of a *relativistic* charged particle in terms of its velocity and the electric and magnetic field intensities. Write down the equation for the following special cases:

(a) Initial velocity is parallel to the electric field and $\mathbf{H} = 0$. Solve the equation to obtain position of charged particle as a function of time.

(b) $\mathbf{E} = 0$ and velocity is transverse to the magnetic field. Obtain the angular frequency as a function of the magnetic field strength and energy of the particle. [6]

7. Consider circular orbits of a charged particle in the Coulomb field generated by an infinitely heavy charged particle, both particles having charge e but of opposite sign. Show that for such orbits

$$v \;=\; \left[\frac{e^2}{mr\gamma}\right]^{1/2} .$$

Here r is the separation of charges, m is the mass of the orbiting charge, v is the speed of the charged particle, $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. Use the fact that $\beta < 1$ to deduce that for such orbits the angular momentum satisfies the inequality $J > e^2/c$. [5]

8. (a) Consider the motion of a charged particle of mass m and charge e in the field of a second charge e'. The mass of the second charge is assumed to be so large that it can be considered to be fixed. Thus, with this assumption, it is clear that the charge e moves in a centrally symmetric electric field with a potential energy α/r where α is ee' in suitable units.

(i) Write down the total energy of the particle in this field. Since the particle moves in a centrally symmetric field, it has a constant angular momentum L. Using polar coordinates (r, ϕ) in the plane of the orbit of the particle write down the total energy E in terms of the radial component of the momentum p_r and the angular momentum L.

(ii) If $\alpha > 0$ then it is clear that the particle can never approach arbitrarily close to the center during its motion. If $\alpha < 0$ (implying attraction), can the charge e fall in toward the center? What is the condition L must satisfy so that e can approach the center? Use the formula for the energy E you wrote down in terms of L and compare the potential energy and kinetic energy terms as $r \to 0$. Can this happen in non-relativistic mechanics?

(iii) The trajectory of the charged particle can be determined using the Hamilton-Jacobi equation. Put $E = -\partial S/\partial t$, $p_r = \partial S/\partial r$ and $p_\phi = \partial S/\partial \phi$ and write down the Hamilton-Jacobi equation. Seek a solution of the form

$$S = -Et + L\phi + f(r)$$

and write down the equation for S. The trajectory is determined by the equation $\partial S/\partial L = constant$. Integrate the expressions obtained for the cases $Lc > |\alpha|$, $Lc < |\alpha|$ and $Lc = |\alpha|$ and thus determine the trajectory for each of these cases.

(iv) In the case of attraction ($\alpha < 0$), for what range of energies can r take finite values and for what range can r go to infinity? If the motion is finite, can the orbit be a closed orbit? Why not? Consider trajectories for which $r \to 0$. What is the shape of the orbit? Is the "fall in" time towards the center finite? [14]

(b) A particle of mass m and charge e with an arbitrary velocity v flies past a nucleus of charge Ze with a large impact parameter b so that the deflection is small. Show that the deflection θ suffered by the particle is given by the expression

$$\theta = \frac{2Ze^2}{mb\beta^2}(1 - \beta^2)^{1/2}$$

where $\beta = v/c$. Note that $\theta \to 0$ when $\beta \to 1$. [14]

Note: The corresponding deflection due to gravity, according to general relativity is

$$\theta = \frac{2M}{b\beta^2}(1 + \beta^2)^{1/2}.$$

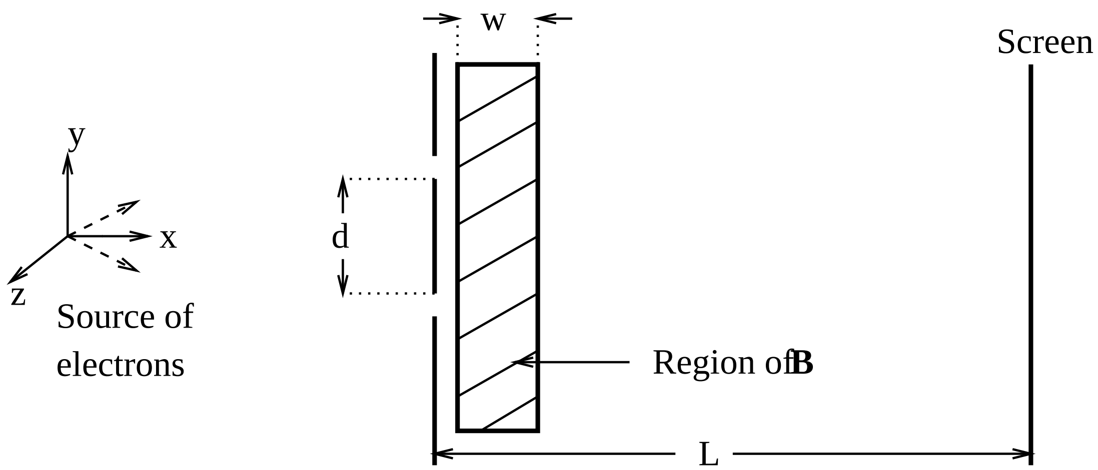
Here the particle m flies past a mass M and is deflected by it. When the particle is moving with the speed of light, the deflection is found to be twice the Newtonian value.

9. In an inertial frame S there is an electromagnetic field $(\mathbf{E}\ ,\ \mathbf{B})$ such that

$$(\mathbf{B}\cdot\mathbf{E})^2 + (\mathbf{B}^2 - \mathbf{E}^2)^2 \neq 0.$$

Show that there exists a Lorentz transformation to another frame S' in which $\mathbf{E}'$ and $\mathbf{B}'$ will be parallel $(\mathbf{E}\times\mathbf{B} = 0)$. [9]

10. In the class we discussed the two slit interference experiment for electrons. It was shown that in the presence of a magnetic field, the phase-difference between two paths is proportional to the magnetic flux through the area enclosed by two paths. Now consider the following arrangement:



There is a uniform magnetic field $\mathbf{B} = B\,\hat{\mathbf{z}}$ localised in a strip of width w $(w \ll L)$.

(a) Show that because of this field, the central maximum of the fringe pattern shifts by an amount

$$\triangle\, y \quad = \quad \frac{q}{\hbar} BwL \left(\frac{\lambda}{2\pi}\right)$$

where λ is the wavelength of the electrons.

(b) Now consider " classical electrons" moving in the $x - y$ plane with an initial velocity $\mathbf{v} = v\mathbf{x}$. Show that the magnetic field $\mathbf{B}$ deflects an electron by an amount $\alpha' = qwB/p$, where p is the momentum of the electron.

Do you see any correspondence between the results in parts (a) and (b)? [14]

11. Let ε^{ijkl} be the totally antisymmetric tensor in four dimensions as defined in class. With respect to rotations, the components of ε^{ijkl} behave like those of a tensor. But, if the sign of one or three of the co-ordinates is changed, the components of ε^{ijkl} do not change sign wheras some of the components of a tensor should change sign. Hence ε^{ijkl} is a "pseudotensor". Pseudotensors of any rank behave like tensors under all co-ordinate transformations except those that cannot be reduced to rotations, i.e. reflections, which are changes in sign of the co-ordinates. Show the following:

(i) $\varepsilon^{iklm}\varepsilon_{prlm} \;=\; -2(\,\delta^i_p\,\delta^k_r \;-\; \delta^i_r\,\delta^k_p\,)$

(ii) $\varepsilon^{iklm}\varepsilon_{pklm} \;=\; -6\delta^i_p$

(iii) $\varepsilon^{iklm}\varepsilon_{iklm} \;=\; -24$

(iv) $\varepsilon^{prst}A_{ip}A_{kr}A_{ls}A_{mt} \;=\; -A\varepsilon_{iklm}$ where A is the determinant formed from the quantities A_{ik}.

(v) $\varepsilon^{iklm}\varepsilon^{prst}A_{ip}A_{kr}A_{ls}A_{mt} \;=\; 24A$ where A is defined as in (iv). [9]

12. The dual of a tensor is defined as

$$F^*{}_{ij} \quad = \quad \frac{1}{2}\epsilon_{ijkl}\,F^{kl}.$$

Here ϵ_{ijkl} is the completly antisymmetric tensor of rank four in four dimensions. Show that

1) $\mathbf{E} \rightarrow -\mathbf{H}$ and $\mathbf{H} \rightarrow \mathbf{E}$ under the "duality transformation" $F \rightarrow F^*$.

2) The product $F\,F^*$ can be written as

$$F^{ij}\,{F^*}_{ij} \;=\; 2\,\frac{\partial}{\partial x^i}\left[\epsilon^{ijkl}\,A_j\,\frac{\partial A_l}{\partial x^k}\right].$$

[14]

13. Consider a charged spatial harmonic oscillator placed in a constant uniform magnetic field. If ω_o is the natural frequency of this oscillator and $\omega_H = eH/mc$, where symbols have their usual meaning. Solve for the frequency of vibration of the charged spatial oscillator. Write down the approximate form for the two limiting cases — a weak magnetic field and a strong magnetic field. When can a magnetic field be considered weak or strong (for this problem!) ? [Assume $v \ll c$].

Write down the expression for $\mathbf{r}(t)$ for a charge that is initially at $x = x_o,\ y = 0,\ z = 0,\ v_x = 0,\ v_y = v_0$ and $v_z = 0$ with the magnetic field along the z axis. What is the maximum distance from the origin that this charge can attain ? Can you describe the motion (qualitatively) in the two limiting cases of weak and strong magnetic field ? [14]

1* A certain piece of elastic breaks when it is stretched to twice its unstretched length. The two ends of the unstretched elastic, connected to identical spaceships, are accelerated (at some time $t = 0$) along the elastic with constant proper acceleration g. Will the elastic break ? If yes, then when will it break.

2* A rod of length l is sliding over a grid that has holes of size $0.5l$ in it. The rod is moving at a speed of $.9c$, will it fall through the grid or not ?

3* What is the minimum energy at which cosmic ray protons can begin to create elecron–positron pairs by interaction with photons of the CMBR ? What fraction of CMBR photons can produce elecron–positron pairs by interacting with a cosmic ray proton of given energy ?
