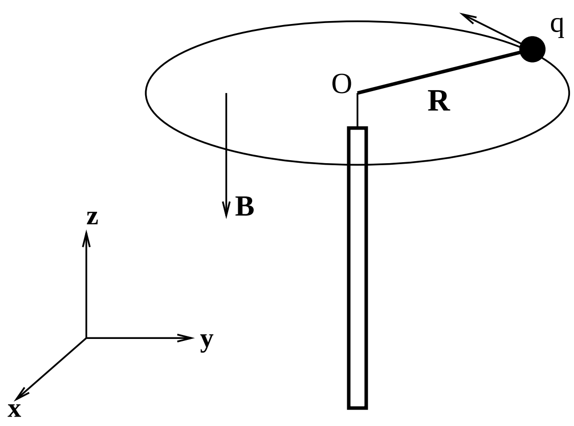
Problem Set 2

<u>To be returned before 11:30 am on October 16</u>

1. An object of fixed mass m and charge q is tethered to a fixed point O by a cord of constant length R. The mass is going around in a circular orbit with a speed v_0, and the cord provides the necessary centripetal force. Now we switch on a magnetic field slowly in the negative z direction. The field is uniform over the whole region at any given instant of time.



(a) Find the electric field produced by this changing magnetic field at any given instant of time.

(b) This electric field acts on the object to change its speed. If the final magnetic field is B_0, find the net change in speed.

(c) Find the magnetic moment (defined as current times area) of the object induced by the magnetic field.

(d) The above picture is often used in text books to calculate the induced magnetic moment of an electron orbiting in a magnetic field. As $\mathbf{B}$ builds up, the orbital angular momentum $\mathbf{L} = mv\mathbf{R}$ varies continuously. But according to quantum theory,

L should be an integral multiple of $\hbar$. How could it change without violating this law? [4]

2. Consider the action for a particle coupled to a scalar field $\phi(\mathbf{x}, t)$

$$\mathcal{A} = -mc\int ds - \frac{\alpha}{c}\int \phi\, ds + \frac{1}{2c}\int d^4x\, \partial^i\phi\, \partial_i\phi$$

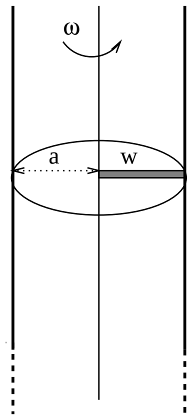
where α is a constant. The second term on the RHS represents the interaction of the field ϕ with the particle (analogous to the $A_i dx^i$ term in electromagnetism) while the third term is the action for the field only (analogous to the $F_{ik}F^{ik}$ term for the electromagnetic field).

(i) By varying the action with respect to the particle trajectories $x^i(s)$, obtain the equation of motion of the particle.

(ii) By varying the action with respect to ϕ obtain the field equations.

(iii) Find the energy momentum tensor for the scalar field using any of the definitions learnt in class. [8]

3. Consider an infinitely long, thin cylinderical shell of radius a with a positive surface charge density σ, rotating with an angular speed ω.



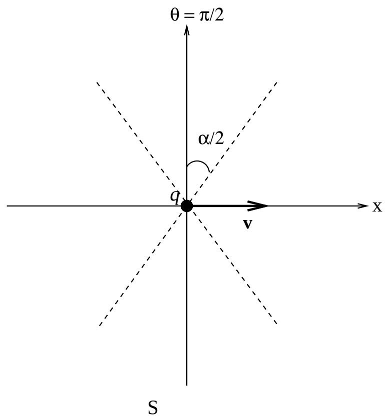
(a) What is the magnetic field inside the cylinder?

(b) Now suppose that we keep a short piece of wire w perpendicular to the axis of the cylinder and extending from the axis out to the surface. The wire is fastened to the cylinder so that it rotates with it. Argue that the end of the wire at the axis picks up a negative charge and by measuring the amount of charge accumulated, we can measure the speed of rotation of this system.

(c) Now put yourself in a frame of reference rotating with the cylinder. Then there is just a charged cylinder at rest. There are no electrostatic forces inside and no forces pushing the charges to the center. How do you reconcile these facts with the result in (b)? Or is the above argument wrong? [6]

4. (a) For the field of a uniformly moving charge, is $\nabla \times \mathbf{E} = 0$? If so, how, and if not, why?

(b) Let the rest frame of a uniformly moving charge q be denoted by S' and let it move in the x direction in the lab frame S. In S' the flux of electric field lines through the surface of a sphere centered at the charge is uniform. Now, as a rough measure of the relativistic "flattening" of the configuration of the electric field lines from a moving charge (as seen in S), we might use the angle α between two conical surfaces which include between them half the total electric flux. That is, half the flux through a sphere shall be contained in the equatorial zone between $\theta = \pi/2 + \alpha/2$ and $\theta = \pi/2 - \alpha/2$.



Consider only the extreme relativistic case with $\gamma \gg 1$. Then it is clear that only angles θ such that $\theta = \pi/2 \pm \epsilon$ with $|\epsilon| \ll 1$ need be considered. Show first that the electric field of a uniformly moving charge can be approximated in this case as

$$\mathbf{E} = \frac{q}{r^2} \frac{\gamma}{(1 + \gamma^2 \epsilon^2)^{3/2}} \hat{\mathbf{r}}.$$

Find α by letting ϵ range from $-\alpha/2$ to $\alpha/2$ and integrating to obtain the flux through the narrow equatorial belt. [4]

5. (a) Consider a uniformly charged spherical shell with charge q and radius a and place a magnetic dipole of strength $\mu\,\hat{\mathbf{n}}$ at the center where $\hat{\mathbf{n}}$ is a unit vector in the direction of the magnetic moment. Show that the angular momentum of the electromagnetic field is

$$\mathbf{L} = \frac{2}{3}\frac{q}{c^2 a}\,\mu\hat{\mathbf{n}}$$

(b) The electron has a magnetic moment $\hbar e/2m$. If we were to think of the electron as a uniformly charged sphere (with a classical radius $R = e^2/2mc^2$) spinning about some axis, what maximum surface velocity would correspond to this magnetic moment? The quantity $c\hbar/e^2$ has the value 137. Make any comments you feel suitable.

(c) Assume that an electron is a uniformly charged sphere of radius R. If it moves at a uniform speed $v \ll c$, show that the associated field momentum is given by

$$\mathbf{p} = \frac{4}{5}\frac{e^2}{Rc^2}\,\mathbf{v}$$

[12]

6. A conducting sphere of radius a and mass m, hanging by an electrically conducting wire of length l from an earthed support, is in equilibrium midway between two vertical parallel earthed metal plates a distance $2b$ apart. The plates are then given potentials $+V$ and $-V$ with respect to earth. Show that, if V is large enough, the sphere will be attracted to one or other of the plates, and calculate approximately the least value of V for which this happens. Assume $b \gg a$. [10]

7. A superconductor is effectively a perfect diamagnetic material, so that $B = 0$ everywhere inside it. A small bar magnet with a weight of $3\,gms$ and a magnetic moment of $0.1\,Am^2$ is lowered towards a horizontal superconducting plane. At some height above the plane the magnet floats freely. Estimate this height. Treat the surface of the superconductor as a mirror and calculate the force between the magnet and the superconductor using the method of images. [6]

8. The "gun" of a CRT consists of a small hot filament F at $x = 0$ and an aperture at $x = d$. Between $x = 0$ and $x = d$ a uniform electric field E accelerates the electrons to velocities which are non-relativistic but are considerably larger than their initial thermal velocities. Show that when they arrive at the plane $x = d$ they appear to have travelled in straight lines (*i.e.* in field free space) from an "image" source I at $x = -d$.

A uniform magnetic field B is now applied axially. Is it still valid to regard I as an image of F? [10]

9. A total charge $-q$ is spread uniformly on the surface of a thin insulating spherical shell of radius a. Thus there is no internal field. A point charge $+q$ is placed at the centre of the sphere. There is now no external field. The point charge can be displaced a distance $r < a$ from the centre without the expenditure of any work. However, this gives the system a dipole moment qr with a corresponding external field. How do you explain the apparent paradox of the appearance of this external field energy? [8]

10. A dust particle in an ionised gas acquires an electric charge, even when the gas is electrically neutral overall, because the numbers of electrons and ions striking the particle each second are not equal. Consider a spherical dust particle of radius r which is placed in ionised hydrogen gas with a temperature T and a density of n protons and n electrons per unit volume. Derive expressions for

(a) the initial rate of accumulation of charge on the particle,

(b) the maximum potential reached at the surface of the particle,

and

(c) the time scale for this maximum potential to be reached. [10]

11. The Action for a non-relativistic charged particle moving in a magnetic field is

$$\mathcal{A} = \int dt\, \frac{1}{2} m\mathbf{v}^2 + \frac{e}{c} \int dt\, \mathbf{A} \cdot \mathbf{v}$$

where e is the charge of the particle, m is its mass and $\mathbf{A}$ is the three vector potential. If the magnetic field were constant, the particle would execute circular motion with frequency $\omega_L = e|\mathbf{B}|/mc$ and depending on the initial conditions it would move with a constant velocity along the direction of the magnetic field. If $\mathbf{B}$ were not constant but varying "slowly" *i.e.* the variation in time is small compared to the time period of the circular motion and the variation spatially is small compared to the radius of the circular motion, then, one would expect the motion to consist of a rapidly circulating motion superposed on an average "drift" motion. The aim of this problem is to determine an effective lagrangian describing this drift motion. To do this, write the position vector $\mathbf{r}$ as

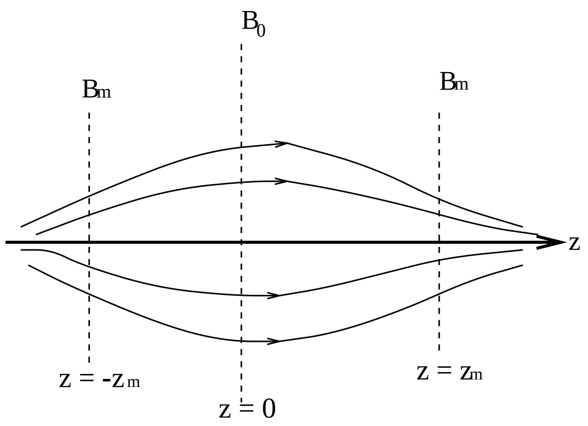
$$\mathbf{r}(t) = \mathbf{R}(t) + \mathbf{q}(t)$$

where $\mathbf{R}(t)$ is the vector to the center of the circle described by the particle at time t and $\mathbf{q}(t)$ is the vector describing just the circular motion. Taylor expand the vector potential about $\mathbf{R}(t)$ keeping only terms of first order in $\mathbf{q}(t)$. Averaging the action over one time period, show that the effective action is given by the expression,

$$\langle \mathcal{A} \rangle = \int dt\, \frac{1}{2} m \dot{\mathbf{R}}^2 + \frac{e}{c} \int dt\, \mathbf{A}(\mathbf{R}) \cdot \dot{\mathbf{R}} + \int dt\, \mu B + \int dt\, \frac{1}{2}\, m v_\perp^2$$

where $\mu = e v_\perp^2 / 2\omega_L c$ and $B = |\mathbf{B}|$. [6]

12. Consider a symmetric magnetic field of the type shown below, which is quasi-homogeneous.



Assume that the axial magnetic field is given by

$$B(z) = B_0 \left[1 + (z/a)^2\right]$$

where B_0 and a are positive constants.

(i) Assuming the motion to be non-relativistic, show that the average force on the particle guiding centre, parallel to the z-axis is

$$\langle \mathbf{F}_{\|} \rangle = - |\mathbf{M}| \frac{\partial B}{\partial z} \hat{\mathbf{z}}.$$

The guiding centre is the centre of the circular orbit for a particle in a constant $\mathbf{B}$ field. $|\mathbf{M}|$ is the magnetic moment of a particle due to circular orbits in such a field. Hence show that the particle executes simple harmonic motion with period given by

$$T = 2\pi a \left(\frac{m}{2\, |\mathbf{M}|\, B_0} \right)^{1/2}$$

(ii) Show that the z-component of the particle velocity is given by

$$\mathbf{v}_{\|} = \left(\frac{2\, |\mathbf{M}|\, B_0}{m} \right)^{1/2} \left[(z_m/a)^2 - (z/a)^2\right]^{1/2}$$

where $\pm z_m$ are the points of magnetic reflection.

(iii) A field configuration like this is known as a magnetic bottle and can be used for the confinement of hot plasma. But particles can "spill" out of a magnetic bottle as well. Now suppose that at the midpoint of the magnetic bottle a particle moves in a helical orbit with the pitch angle α_0. Find, as a function of energy and magnetic moment of the particle, the range of α_0 for which the particle will remain confined in the bottle.

[15]

13. (i) Consider any system whose action integral has the following form

$$\mathcal{A} = \int \Lambda \left(q, \frac{\partial q}{\partial x^j} \right) d^4 x$$

where Λ is called the "Lagrangian density" and is an arbitrary function of the co-ordinates q and of their first derivatives with respect to space and time. Using this lagrangian density and the definition of the energy momentum tensor in terms of the lagrangian given in class, calculate T^{ik} explicitly.

(ii) The definition of T^{ik} is not unique. If one were to add a term to T^{ik} of the form

$$T^{ik} = T^{ik} + \frac{\partial}{\partial x^l}\psi^{ikl}, \quad \psi^{ikl} = -\psi^{ilk}$$

prove that the divergence of T^{ik} is still zero. Does the total four momentum of the system change? Define the four tensor of angular momentum in the following manner.

$$L^{ik} = \int (x^i\, dP^k - x^k\, dP^i) = \frac{1}{c}\int (x^i\, T^{kl} - x^k\, T^{il})\, dS_l$$

If the angular momentum is to be conserved, what condition must the energy momentum tensor satisfy?

(iii) As a special case, calculate the energy momentum tensor for the electromagnetic field using the lagrangian density,

$$\Lambda = -\frac{1}{16\pi c} F_{kl}\, F^{kl}$$

Is T^{ik} calculated from this lagrangian density symmetric? If not, how does one symmetrize it? Does the symmetrized tensor match the expression derived in class?

(iv) When charges and currents are present, prove using Maxwell's equations that

$$\partial_k T_i^k = -\frac{1}{c} F_{ik} j^k$$

where the symbols have their usual meaning. [5]