Problem Set 3

To be returned before 11:30 am on Nov.27

General Instructions

The questions below have a total weightage of 122 marks and your score will be treated as one obtained out of 100. For instance, if you get a score of 85 out of 122, we will actually treat it as 85/100 . All scores above 100 will be set to 100.

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1. A very large number $N + 1$ of polaroids are arranged in a sandwich. The angle of the easy axis of each polaroid is a constant angle α greater than that of its immediate predecessor in the sandwich. Thus the last polaroid is at an angle $\theta = N\alpha$ from the first. Neglecting any losses due to reflection at the many surfaces, and supposing that linearly polarized light of intensity I_0 is incident on the first polaroid with its polarization along the easy axis of the first polaroid, find the output intensity. Take N to be very large and retain only the first interesting terms in an appropriate power series.

One can therefore see that even if θ is 90 degrees, so that the first and last polaroids are crossed, the output intensity is (almost) equal to the input intensity if we have enough intermediate polaroids. We can "gently" rotate the plane of polarization and lose nothing! Thus, if we cemented a large number of polaroids together (using clear cement with the same index as the polaroids, so as to minimize reflections), we would have something like a gigantic sugar molecule, which rotates the plane of polarization without absorbing any energy. Does this sound reasonable? [5]

2. A point object at distance p is focused to a point image on a camera film located at a distance q behind a thin lens of diameter D. Another point object at distance $p + \Delta p$ will not be in focus on the film. It will pass through its focus before (or after) reaching the film and will give a "circle of confusion" on the film.

(a) Show that the diameter d of the circle of confusion on the film of an out-of-focus point is given by $d \approx D(q/p^2)\Delta p$. Thus for a given "tolerable" circle of confusion, *i.e.*, a given value of d, and for given q and p, "the depth of focus" Δp is inversely proportional to the lens diameter D. Small D gives large depth of focus. The focal length divided by the diameter is called the "f-number". Thus large f-number means small diameter of the aperture "stop" on the lens and gives large depth of focus. For D equal to zero, we have a "pinhole camera"; and this formula would say that the depth of focus is infinite.

(b) If D gets too small, we cannot neglect diffraction. Show that diffraction gives a circle of confusion $d \approx q\lambda/D$. Now suppose that you are limited by the photographic film's "grain density" or anything else about the film. Also, suppose you are not limited by intensity (which might require large D). Define d_{av}^2 as the sum of the squares of the two contributions to d, one from depth of focus and one from diffraction. Minimize d_{av}^2 as a function of the lens diameter D, holding everything else constant. Show that, for given λ, p, and Δp, the least fuzzy image is obtained with D for which $D^2 = \lambda p^2/\Delta p$.

(c) Forget about diffraction. Suppose you are photographing two people at once; one is 15 feet away and the other is 25 feet away. The lens focal length is 10cm. You want the circle of confusion on *both* people to be less than 1mm in diameter "in the object space" *i.e.*, 1mm "on the person". Find the required f-number. Use rough approximations at your discretion (for example, one can take $p \approx$ 20ft).

(d) Does diffraction make the image noticeable worse in the geometry of part (c)? [8]

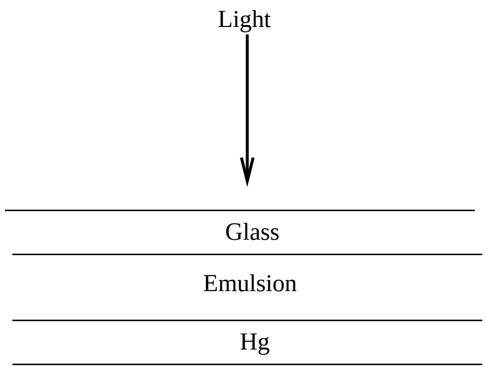
3. Determine the force exerted on a wall from which an incident plane electromagnetic wave is reflected (with reflection coefficient R). Assume that the angle of incidence is θ. [5]

4. Explain the following observations.

(a) A film of a horse race, taken using a telephoto lens, shows the horses strangely foreshortened as they gallop towards the camera. (A telephoto lens, like a telescope, makes far objects appear near by giving rise to an angular magnification. Approximate a horse by a rectangular body with a vertical leg at each corner. Let the seperation of the forelegs (and hind legs) be about 0.5m and distance between the front and hind legs be about 2m.)

(b) A photograph of skyscrapers taken with a wide-angle lens shows them distorted into barrel-like columns. (Hint: In order to reduce image defects such as astigmatism and field curvature in a camera lens, an aperture stop is often placed on the optic axis a little way in front of the lens. Assume that the wide-angle lens has such an aperture stop in front of it.) [15]

5. A method of color photography was invented by Lippmann in 1881. The photographic plate consists of an extremely fine–grain emulsion on a glass plate which is covered on the emulsion side with a layer of mercury to form a reflecting surface. It is exposed in a camera with the glass side toward the light. After development and with the layer of mercury in place, the plate is illuminated with white light and viewed by reflection. A faithfully colored picture of the image is seen. Explain what is happening.



[15]

6. (a) Find the index of refraction of Aluminium for X-rays of wavelength 1.56×10^{-8}cm. Assume that all the electrons in Aluminium have natural frequencies very much less than the frequency of the X-rays.

(b) The index of refraction of the ionosphere for radio-waves of frequency 100 MHz is $n = 0.90$. Find the density of electrons per cubic centimeter in the ionosphere.

(c) How much is the blue light ($\lambda = 4500 A$) of the sun attenuated in going through the atmosphere when the sun is (i) at the zenith, and (ii) 10^o from the horizon?

(d) Find the r.m.s. electric field above the earth's atmosphere (in volts/cm), due to sunlight. The solar constant (defined for a point above the earth's atmosphere) is $1.94 \mathrm{cal/cm^2/min}$ ($1\mathrm{cal} = 4.18\mathrm{J}$). [5]

7. Suppose that you are out on a hot summer day and the air near the pavement is hotter by an amount ΔT compared to its temperature T above the pavement. Assume as an approximation that the temperature change is sudden. Let ϕ be the angle of incidence of a ray at the critical angle for total internal reflection, with ϕ measured from the pavement *i.e.* ϕ is 90^o minus the angle of incidence as measured from the normal to the pavement. Assuming $(n-1) \ll 1$, derive the formula

$$\phi \approx \left(\frac{2(n-1)\Delta T}{T}\right)^{1/2} \quad \text{for } \phi \ll 1$$

Assume that in the region under consideration, the density ρ of the air goes as

$$\rho = \rho_0 + \frac{\text{constant}}{T}$$

[5]

8. (a) It is known that the primary rainbow is formed when the sunlight incident on the water drops in the air undergoes reflection and subsequently reaches the ground observer. Assume that the sunlight incident on the drop is parallel to the ground and find out at what angle above the ground is the rainbow formed in the sky (refractive index for water is 4/3).

(b) What is the angular width of the rainbow? (refractive index for blue light is 1.343 and that for red light is 1.331)

(c) Is the sunlight coming out of the rainbow polarized? Ir so, what is the plane of polarization? [8]

9. (a) Suppose you have a diffraction limited laser beam of diameter $D = 2$mm, with wavelength 6000A. How much does the beam diameter increase in a distance of 50 metres?

(b) The angular resolution limit of the human eye is about 1/2000. The diffraction limit of angular resolution for the human eye is given by the angular size of the image spot on the retina from an incident plane wave emitted by a distant point source. Take the pupil diameter to be 2mm and calculate the diffraction limit for light of 5500A wavelength. How does it compare with the actual angular resolution limit? [8]

10. It is often said that the travelling wave from a distant point source is "like" a plane wave over a "limited region" transverse to the line of sight from the point source to the field point. How limited is this region? Suppose the source is at a distance L and we wish to consider a circular plane region of radius R transverse to the line of sight from the source. How large can R be so that the phase at the centre of the circle and that at the edge of the circle differ by less than $\Delta\phi$ radians? [5]

11. (a) A standard result in elementary physics is to calculate the theoretical distance to the visible horizon on a smooth spherical earth, ignoring any effect of the atmosphere on the path of light rays. Show that if the observer is at a height h above the surface of the earth, then, the distance D to the horizon is given by the relation

$$D(h) = \sqrt{2Rh}$$

where R is the radius of the earth and $h \ll R$.

(b) We now take into account the fact that the atmosphere is a refracting medium. Assume that the refractive index varies with the height z above the earth as

$$n(z) = 1 + ke^{-\alpha z}$$

where $k = 2.77\times10^{-4}$ and $\alpha \approx 0.105\text{km}^{-1}$. You may also assume that $n(z=0) = 1$. Under the approximation of "small bending" and "small altitude", show that the corrected distance to the horizon is given by

$$D(h) = \sqrt{2Rh}\left(1 + \frac{R}{2}k\alpha\right)$$

Is the correction significant? [5]

12. The ionosphere can be considered as an ionized medium containing N essentially free electrons per unit volume. Show that if a linearly polarized wave propagates in the ionosphere in a direction parallel to that of the small uniform magnetic field H produced by the earth, its plane of polarization will be rotated through an angle proportional to the distance travelled by the wave. Calculate the constant of proportionality. You may ignore the magnetic field of the travelling wave in comparision with that of the earth's field. [15]

13. Polarized light is passed parallel to the axis through a solid glass cylinder of length L which is rotating with an angular velocity Ω around its axis. Find the rotation of the plane of polarization. Assume a constant index of refraction n and permeability of 1.

This problem was first treated (incorrectly) by J. J. Thomson; then (correctly) by E. Fermi in *Rend. Lincei.* **32** (I) (1923). Professor Fermi put the problem on the 1948 "Basic" exam of Chicago University as a library problem. One student located Fermi's original paper, translated it, and was the only student to give a correct solution.

The problem may be tackled as follows: Consider Maxwell's equations in a medium with no charges and currents. If $\mathbf{P}$ is the polarization in the rest frame of the cylinder, then to first order in v/c, one has

$$\mathbf{P} = \left(\frac{\epsilon - 1}{4\pi}\right)\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right) \quad \text{and} \quad \mathbf{M} = \frac{\mathbf{P}\times\mathbf{v}}{c},$$

with

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} \quad \text{and} \quad \mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}.$$

Therefore to the first order in v/c write down relations for $\mathbf{E}$ and $\mathbf{H}$ in terms of $\mathbf{D}$ and $\mathbf{B}$. The relations obtained are the non-relativistic limit of Minkowski's equations for moving media. Assume the axis of rotation to be along $\hat{\mathbf{z}}$ and look for circularly polarized solutions for $\mathbf{D}$ and $\mathbf{B}$ with $D_z = B_z = 0$. Substitute these solutions in Maxwell's equations and using the fact that $\mathbf{v} = \mathbf{\Omega} \times \mathbf{r}$ obtain the required solution for $\mathbf{E}$. [15]

14. Consider nestled spherical and parabolic mirrors as shown in the figure. Take the $+\hat{\mathbf{z}}$ direction to the right (along the axis of symmetry) and x transverse to z; take $x = z = 0$ at the apex of the mirrors.

(a) Show that the parabolic surface is given by

$$z = \frac{x^2}{4f}.$$

(b) Show that the spherical surface is given (for $x \ll f$) by

$$z = \frac{x^2}{4f} + \frac{x^4}{64f^3} + \ldots$$

(c) Compare a spherical mirror with aperture of diameter D and with focal length f to a parabolic mirror with the same D and f. For the spherical mirror, consider the angular deviation $\delta\theta$ of the "worst" rays (near the rim of the aperture) due to spherical aberration. $\delta\theta$ is the deviation from the $\hat{\mathbf{z}}$ direction for rays from a point source. Show that $\delta\theta$ is less than the diffraction angular width $\Delta\theta \approx \lambda/D$ provided that

$$D < 4f\left(\frac{\lambda}{4f}\right)^{1/4}.$$

Thus, verify, for visible light and for a focal length $f \approx 50$in, that a spherical mirror is about as good as a parabolic mirror provided the mirror diameter D is less than ~ 3.5in. [8]