Problem Set 4

<u>To be returned before 11:30 am on December 21</u>

General Instructions

The questions below have a total weightage of 150 marks and your score will be treated as one obtained out of 100. For instance, if you get a score of 85 out of 122, we will actually treat it as 85/100 . All scores above 100 will be set to 100.

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1. This question asks you to work on a model for a "real-life" situation in astrophysics and is a bit open-ended. You can explore a lot of stuff if you feel like, based on what is given.

The problem has to do with heating and cooling of gas in the interstellar medium. The following steps will help you to develop a model, work out the details numerically and understand the results. You get credit for all parts: developing the model, investigating it and interpreting the results.

Consider a gas cloud containing "hydrogen", "oxygen" and "carbon", with "....." signifying that we will drastically simplify their structure in our model. The gas is permeated by a flux of ionising photons with a power law frequency dependence. This will be a heating source for the gas. The electrons in the ionised gas will hit the atoms and excite them to higher levels. When the atoms drop back to ground state *radiatively*, the energy goes into photons. Thus, this process — of collisional excitation followed by radiative de-excitation — can cool the gas. In addition, the Compton interaction between the photons and electrons can also act as a cooling/heating agent.

When all these processes reach a steady state, the number of atoms in different excited states will be time independent and cooling will balance the heating. These conditions will determine the equilibrium properties of the gas. Your task is to understand this situation more quantitatively using a simple model developed below:

(1) Let the ionizing flux of photons have the spectrum

$$\begin{aligned} F(E)dE &= F_{\rm H}\left(\frac{E}{\chi_{\rm H}}\right)^{-s} dE && (\chi_{\rm H} < E < 10^4\chi_{\rm H}) \\ &= 0 && (\text{otherwise}) \end{aligned}.$$

Here $F(E)dE$ has the dimensions $erg\, cm^{-2}\, s^{-1}$. You can take $s = 1.5$ and $\chi_{\rm H} = 13.6\, eV$. The quantity $F_{\rm H}$ is kept unspecified at the moment.

(a) Calculate the total flux $F_{\rm tot}$ and mean energy of the photons $\langle E\rangle$.

(b) Calculate the quantity

$$P \equiv \frac{1}{c}\int_0^\infty F(E)dE$$

where c is the speed of light. What does this quantity represent?

(2) The cloud is assumed to be optically thin to all relevant photons. We will also assume the densities to be spatially uniform. [So we don't need to do any radiative transport!]. It is populated by the three kinds of atoms with the following properties:

(a) "hydrogen" : This has two levels, a ground state and an *ionised* state separated by 13.6 eV. The ionisation cross section is

$$\begin{aligned} \sigma_{\rm H} &= \sigma_0\left(\frac{E}{\chi_{\rm H}}\right)^{-\mu} && (\text{for } E > \chi_{\rm H}) \\ &= 0 && (\text{otherwise}) \end{aligned}$$

with $\sigma_0 = 6\times 10^{-18}\, cm^2$ and $\mu = 2.7$. [Can you justify the order-of-magnitude of σ_0?]. The rate of recombination per unit volume is $\alpha_{\rm H} n({\rm H}^+) n_e$ with

$$\alpha_{\rm H} = 10^{-13} cm^3\, s^{-1}\left(\frac{T_e}{10^4\, K}\right)^{-1/2}$$

[Can you justify the T_e dependence of $\alpha_{\rm H}$?] Argue that the recombination cooling rate is

$$\left(\frac{dE}{dtdV}\right)_{\rm cool,\, rec} = \alpha_{\rm H} n_e n({\rm H}^+)(k_B T_e)$$

while the photoionisation heating rate is

$$\left(\frac{dE}{dtdV}\right)_{\rm heat,\, ion} = n({\rm H}_0)\int_{\chi_{\rm H}}^\infty (E - \chi_{\rm H})\sigma_{\rm H}(E)\frac{F(E)}{E}dE$$

where $n(\mathrm{H_0})$ is the number density of neutral hydrogen. [What are the implicit assumptions you are making?] Compute this expression. Write an equation expressing the photoionization equilibrium of the number of atoms in the two states.

(b) "oxygen": This atom has two bound energy levels separated by $2.5\, eV$ and is connected by a "forbidden" radiative transition with a spontaneous decay rate of $A_{21}(\mathrm{O}) = 0.01\, s^{-1}$. The collisional excitation/de-excitation rates are

$$\begin{aligned} C_{\mathrm{oxy}}(1 \to 2) &= \gamma n_e \exp\left(-\frac{E_{\mathrm{gap}}(\mathrm{O})}{k_B T_e}\right) \\ C_{\mathrm{oxy}}(2 \to 1) &= \gamma n_e \end{aligned}$$

with $E_{\mathrm{gap}}(\mathrm{O}) = 2.5\, eV$ and

$$\gamma = 10^{-7} cm^3\, s^{-1} \left(\frac{T_e}{10^4\, K}\right)^{-1/2}$$

Can you justify the ratio $C_{\mathrm{oxy}}(1 \to 2)/C_{\mathrm{oxy}}(2 \to 1)$ and the T_e dependence of γ? [The numbers are modelled after the OIII 5007,4959 line, just in case you want to know]. Write an equation expressing the equilibrium of the number of atoms in the two levels.

(c) "carbon": This atom has two bound levels connected by a permitted electric dipole transition with a spontaneous decay rate of $A_{21}(\mathrm{C}) = 10^8\, s^{-1}$. [Can you explain why $A_{21}(\mathrm{C})$ and $A_{21}(\mathrm{O})$ are widely different?] The collisional excitation rates are the same as that for oxygen with $E_{\mathrm{gap}}(\mathrm{C}) = 8\, eV$ replacing $E_{\mathrm{gap}}(\mathrm{O})$. [This is modelled after the CIV 1549 line]. Write an equation governing the equilibrium between the two levels.

(3) We also need to consider the inverse Compton heating/cooling between the electrons and the ionising photons. Show that the Compton heating/cooling rate per unit volume is given by

$$\left(\frac{dE}{dtdV}\right)_{\mathrm{comp,\, cool}} = n_e \left(\frac{\sigma_{\mathrm{Thom}} F_{\mathrm{tot}}}{m_e c^2}\right) (4 k_B T_e - \langle E \rangle)$$

Express this in terms of the χ_{H}, F_{H} and s.

(4) The thermal equilibrium of the cloud is decided by the equation heating=cooling where you should include the photoionisation heating, recombination cooling, radiative cooling

of oxygen and carbon and inverse Compton heating/cooling. Express the equations in terms of the following variables: For the composition, use

$$\begin{aligned}
x_{\rm H} &\equiv \frac{n({\rm H}^+)}{n({\rm H})} = \text{fraction in excited state} \\
x_{\rm O} &\equiv \frac{n_2({\rm O})}{n({\rm O})}; \quad x_{\rm C} = \frac{n_2({\rm C})}{n({\rm C})} = \text{fraction in excited state} \\
r_{\rm O} &\equiv \frac{n({\rm O})}{n({\rm H})} = \text{relative abundance of oxygen} \\
r_{\rm C} &\equiv \frac{n({\rm C})}{n({\rm H})} = \text{relative abundance of carbon}
\end{aligned}$$

Let the pressure of the gas be

$$P_{\rm H} = n({\rm H})(1 + x_{\rm H})k_BT_e$$

Define the dimensionless ratio $f = (P/P_{\rm H})$ which measures [essentially] $(P_{\rm rad}/P_{\rm gas})$ [Why?]. Show that you get the following equations for the abundances:

$$\begin{aligned}
x_{\rm H} &= \left\{1 + \frac{s+\mu}{s-1}\left(\frac{\chi_{\rm H}}{k_BT_e}\right)\cdot\frac{\alpha_{\rm H}}{c\sigma_0}\frac{1}{f}\right\}^{-1/2} \\
x_{\rm O} &= \left\{1 + e^{E_{\rm O}/k_BT_e}\left(1 + \frac{A_{21}({\rm O})}{\gamma}\frac{1}{n_{\rm H}x_{\rm H}}\right)\right\}^{-1} \\
x_{\rm O} &= \left\{1 + e^{E_{\rm C}/k_BT_e}\left(1 + \frac{A_{21}({\rm C})}{\gamma}\frac{1}{n_{\rm H}x_{\rm H}}\right)\right\}^{-1}
\end{aligned}$$

In all these equations, you should express the number density $n_{\rm H} \equiv n({\rm H})$ in terms of the pressure $P_{\rm H}$ as $n_{\rm H} = P_{\rm H}/(1+x_{\rm H})k_BT_e$. [Why ?].

The thermal equilibrium equation is

$$R_{\rm cool}(T_e) - R_{\rm heat}(T_e) = 0$$

with net cooling/heating rate *per baryon* being

$$\begin{aligned}
R_{\rm cool} &= \alpha_{\rm H}x_{\rm H}^2n_{\rm H}k_BT_e + x_{\rm O}r_{\rm O}A_{21}({\rm O})E_{\rm gap}({\rm O}) + x_{\rm C}r_{\rm C}A_{21}({\rm C})E_{\rm gap}({\rm C}) \\
R_{\rm heat} &= \frac{\alpha_{\rm H}x_{\rm H}^2n_{\rm H}\chi_{\rm H}}{s + \mu - 1} + x_{\rm H}P_{\rm H}cf\left(\frac{\sigma_{\rm Thom}}{m_ec^2}\right)[\langle E\rangle - 4k_BT_e]
\end{aligned}$$

(5) Now begins the fun. Take $s = 1.5$, $r_{\rm O} = 10^{-4}$, $r_{\rm C} = 10^{-4}$, $f = 1$ and explore the equlibrium temperature as a function of the pressure for the range $10^4\,cm^{-3}K < (P_{\rm H}/k_B) < 10^{14}\,cm^{-3}K$. Plot the equilibrium temperature as a function of pressure. Evaluate the relative importance of different cooling and heating processes. Can you explain why oxygen cooling dominates at one end and carbon at the other ?

[This will require your writing a short computer program and I have enclosed one I wrote in mathematica. You can use it, if you care to pick up mathematica! But you must understand it and make sure it is correct; I am *not* assuring that!. Or you can write one in your favourite language.]

(6) Take $s = 1.5$, $r_{\rm O} = 10^{-4}$, $r_{\rm C} = 10^{-4}$, $(P_{\rm H}/k_B) = 10^{14}\,cm^{-3}K$ and vary f in the range $10^{-4} < f < 10^4$. Plot the equilibrium temperature as a function of f. Which heating and cooling processes dominate each equilibrium? Which of these equlibria are stable and which are not? What happens if you remove oxygen? carbon? both? [Check all these numerically by running the code]. Can you explain why there is a thermostat in operation, in physical terms?

(7) Extra credit: This simplified model gives results which are reasonable compared to really "professional" codes which take into account a lot more elements, lines etc. Can you use this model or simple variants of it for any further explorations ? [100]

2. Suppose that an observer at rest with respect to the fixed distant stars sees an isotropic distribution of stars. That is, in any solid angle $d\Omega$ he sees $dN = N(d\Omega/4\pi)$ stars, where N is the total number of stars he can see.

Suppose now that another observer (whose rest frame is K') is moving at a relativistic velocity β in the x direction. What is the distribution of stars seen by this observer? Specifically, what is the distribution function $P(\theta', \phi')$ such that the number of stars seen by this observer in his solid angle $d\Omega'$ is $P(\theta', \phi')d\Omega'$? Check to see that $\int P(\theta', \phi')d\Omega' = N$ and check that $P(\theta', \phi') = N/4\pi$ for $\beta = 0$. In what direction will the stars "bunch up" according to the moving observer? [5]

3. Consider a sphere of ionized hydrogen plasma that is undergoing spherical gravitational collapse. The sphere is held at constant isothermal temperature T_0, uniform density and constant mass M_0 during the collapse, and has decreasing radius $R(t)$. The sphere cools by emission of bremsstrahlung radiation in its interior. At $t = t_0$ the sphere is optically thin.

(a) What is the total luminosity of the sphere as a function of M_0, $R(t)$ and T_0 while the sphere is optically thin?

(b) What is the luminosity of the sphere as a function of time after it becomes optically thick?

(c) Give an implicit relation, in terms of $R(t)$, for the time t_1 when the sphere becomes optically thick.

(d) Draw a qualitative curve of the luminosity as a function of time.

[5]

4. An ultrarelativistic electron emits synchrotron radiation. Show that its energy decreases with time according to

$$\gamma = \gamma_0(1 + A\gamma_0 t)^{-1}, \qquad A = \frac{2e^4 B_\perp^2}{3m^3c^5}.$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$ and v is the magnitude of the velocity of the charge. Here γ_0 is the initial value of γ and $B_\perp = B\sin(\alpha)$. Show that the time for the electron to lose half its energy is

$$t_{\frac{1}{2}} = (A\gamma_0)^{-1} = \frac{5.1 \times 10^8}{\gamma_0 B_\perp^2}.$$

How does one reconcile the decrease of γ here with the result of constant γ implied by the fact that the magnetic field does no work on the charge? [5]

5. The spectrum shown in Fig.1 (see page 11) is observed from a point source of unknown distance d. A model for this source is a spherical mass of radius R that is emitting synchrotron radiation in a magnetic field of strength B. The space between us and the source is uniformly filled with a thermal bath of hydrogen that emits and absorbs mainly by bound-free transitions, and it is believed that the hydrogen bath is unimportant compared to the synchrotron source at frequencies where the former is optically thin. The synchrotron source function can be written as

$$S_\nu = A(\mathrm{erg\,cm^{-2}\,s^{-1}\,Hz^{-1}}) \left(\frac{B}{B_0}\right)^{-1/2} \left(\frac{\nu}{\nu_0}\right)^{5/2} .$$

The absorption coefficient for synchrotron radiation is

$$\alpha_\nu^s = C(\mathrm{cm^{-1}}) \left(\frac{B}{B_0}\right)^{(p+2)/2} \left(\frac{\nu}{\nu_0}\right)^{-(p+4)/2} ,$$

and that for bound-free transitions is

$$\alpha_\nu^{bf} = D(\mathrm{cm^{-1}}) \left(\frac{\nu}{\nu_0}\right)^{-3} ,$$

where A, B_0, ν_0, C and D are constants and p is the power law index for the assumed power law distribution of relativistic electrons in the synchrotron source.

(a) Find the size of the source R and the magnetic field strength B in terms of the solid angle $\Omega = \pi(R^2/d^2)$ subtended by the source and the constants A, B_0, ν_0, C and D.

(b) Now using D and ν_1, in addition to the previous constants, find the solid angle of the source and its distance. [5]

6. (a) The 'bound–bound' absorption cross-section $\sigma_{\mathrm{bb}}(\omega)$ for the atom is defined by the relation

$$\begin{bmatrix} \text{fraction of photons} \\ \text{absorbed per second} \end{bmatrix} = \int_0^\infty \sigma_{\mathrm{bb}}(\omega)\,[\text{flux of photons}]\,[\text{density of states}]\,\mathrm{d}\omega$$

or, equivalently,

$$\frac{dP_{if}}{dt} = \int_0^\infty \sigma_{\mathrm{bb}}(\omega)\,[cn(\omega)]\,\frac{4\pi(\omega/c)^2 d(\omega/c)}{(2\pi)^3}$$

where P_{if} is the probability of absorption between states i and f of the atom summed up over all directions $\hat{\mathbf{k}}$ and polarization states. dP_{fi}/dt is given by the formula

$$\frac{dP_{fi}}{dt} = \frac{2}{3}\frac{q^2}{\hbar c^3} n(\omega_{fi})\omega_{fi}^3\,|\mathbf{x}_{fi}|^2$$

where $\mathbf{x}_{fi}$ is the transition element of the position operator (in the dipole approximation). It is convenient to write $\sigma_{\rm bb}$ as

$$\sigma_{\rm bb}(\omega) \;=\; \frac{\pi q^2}{mc} f[2\pi\delta(\omega - \omega_{fi})] \;\equiv\; \frac{\pi q^2}{mc} f\phi_\omega$$

where f is called the "oscillator strength" and ϕ_ω is called the "line profile function". Show that, in the dipole approximation,

$$f \;=\; \frac{2}{3}\frac{m\omega}{\hbar}\,|\mathbf{x}_{fi}|^2 \;=\; \frac{2}{3}\left[\frac{m\omega^2\,|\mathbf{x}_{fi}|^2}{\hbar\omega}\right].$$

Interpret this result. Write the total emission and absorption rates for isotropic, unpolarized radiation as

$$\left(\frac{dP_{\rm abs}}{dt}\right) \;=\; \left(\frac{2q^2}{mc^3}\right) n_\alpha(\omega)\omega^2 f,$$
$$\left(\frac{dP_{\rm emis}}{dt}\right) \;=\; \left(\frac{2q^2}{mc^3}\right) [1 + n_\alpha(\omega)]\omega^2 f$$

where $n_\alpha(\omega) = [n(\omega)/2]$ is the number of photons in each polarization state.

(b) Compute f for the hydrogen atom when the initial state is the $n = 1$ ground state and the final state is any one of the three states $\{n = 2,\, l = 1,\, m = -1, 0 \,\text{or}\, 1\}$. Sum over all the three final states and show that

$$f \;=\; \frac{2^{14}}{3^9} \;\cong\; 0.83.$$

[5]

7. A sufficiently energetic photon can ionize the hydrogen atom, that is, the electron can absorb the photon and make a transition from the $n = 1$ ground state to a free particle state. Let the rate of such transitions be (dP/dt) when the number of incident photons is $n(\omega)$.

(a) The cross-section $\sigma_{\mathbf{PI}}$ for photoionization is defined by the relation

$$\left(\frac{dP}{dt}\right) dN_e \;=\; \sigma_{\mathbf{PI}}(\omega)\cdot cn(\omega)\cdot \frac{4\pi(\omega/c)^2 d(\omega/c)}{(2\pi)^3},$$

where $dN_e = \left[V d^3p/(2\pi\hbar)^3\right]$ is the density of states available for the final electron. Explain the factors in this definition.

(b) To evaluate $\sigma_{\mathbf{PI}}(\omega)$, one needs to compute (dP/dt) which — essentially — involves computing the matrix element of $(\exp(i\mathbf{k}\cdot\mathbf{x})\mathbf{p})$ between the initial and final states of the electron. The initial state is the $n = 1$ ground state of a hydrogen atom with energy $(-E_0)$. Assume , for simplicity, that the final state is a plane wave state. Show that, if $\hbar\omega \gg E_0$, then

$$\sigma_{\mathbf{PI}}(\omega) = \frac{2^8\pi}{3}\left(\frac{q^2}{\hbar c}\right) a_0^2 \left(\frac{E_0}{\hbar\omega}\right)^{7/2},$$

with $a_0 = (\hbar^2/mq^2)$. [15]

8. Reproduced in Fig.2 (see page 11) is Fig.1 from Henriksen and Mushotzky (1986, Astrophysical Journal, **302**, 287). This is the spectrum of radiation from the hot gas in the Coma cluster observed in X-rays (notice the peculiar units used by X-ray astronomers). They claimed that the best fit to the spectrum is obtained by a thermal bremsstrahlung spectrum with $T = 7.6\,keV$. Estimate the flux at $10\,eV$ from the plot. Assuming a distance of $90\,Mpc$, show that the rate of photon emission from the gas at $10\,eV$ is $\sim 10^{51}\,s^{-1}keV^{-1}$. Find the value of $\int n_e^2 dV$. Assume that n_e is uniform (*i.e.* you can take n_e out of the integral) and that most of the radiation comes from within $1\,Mpc$ of the Coma cluster (*i.e.* you can estimate the volume V of the radiating region by assuming a spherical geometry). Estimate n_e. [5]

9. A certain gas emits thermally at the rate $P(\nu)$ (power per unit volume and frequency range). A spherical cloud of this gas has radius R, temperature T and is at a distance d from the earth ($d \gg R$).

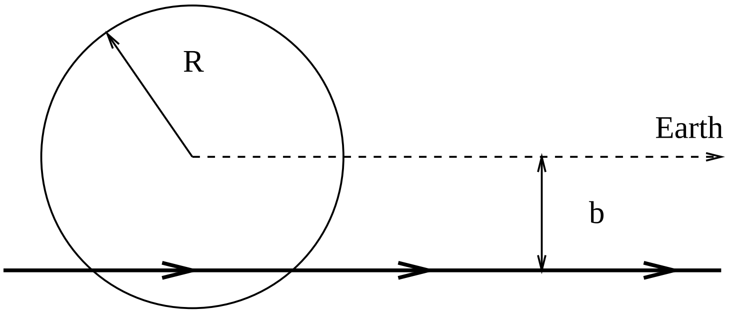
(a) Assume that the cloud is optically *thin*. What is the brightness of the cloud as measured on earth? Give your answer as a function of the distance b away from the cloud center, assuming the cloud may be viewed along parallel rays as shown in the figure.

(b) What is the effective temperature of the cloud?

(c) What is the flux F_ν measured at earth coming from the entire cloud?

(d) How do the measured brightness temperatures compare with the cloud's temperature?

(e) Answer parts (a)–(d) for an optically *thick* cloud.



Detection of rays from a spherical emitting cloud of radius R.

[You should be able to pick up the jargon used in this question from some standard text!]

[5]