Final Exam

10-12-96

To be returned before 7:15 pm

General Instructions

1. The questions below have a total weightage of 120 marks and your score will be treated as one obtained out of 100. For instance, if you get a score of 85 out of 122, we will actually treat it as 85/100 . All scores above 100 will be set to 100.
2. You are expected to solve this paper on your own without consulting any other human being. You may consult any of the books in the library or your class notes, returned assignment sheets etc.
3. Since the aim of the course was to train you to think like an astrophysicist, we expect you to be not too rigid in interpreting the questions and to invoke reasonable approximations and models whenever appropriate.
4. We expect you to take only about 3.5 hours to do this exam; so you have plenty of time. Don't panic!

   Good luck!

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1. (a) Given the three-vector potential $\mathbf{A}$ with $\nabla \cdot \mathbf{A} \neq 0$, find a gauge function $f(\mathbf{r})$ such that upon performing a gauge transform to a new vector potential $\mathbf{A}'$, $\nabla \cdot \mathbf{A}' = 0$. Give a formula for $f(\mathbf{r})$ in terms of $\mathbf{A}(\mathbf{r})$.

(b) Prove that,

$$\nabla^2 \left(\frac{1}{|\mathbf{r}|} \right) = -4\pi \delta_{\mathrm{D}}(\mathbf{r})$$

where δ_{D} is the Dirac Delta function.

(c) What charge distribution gives the spherically symmetric potential

$$V(r) = \frac{e^{-\lambda r}}{r} \qquad \text{where} \quad r = |\mathbf{r}|$$

where λ is a positive constant?

[12]

2. (a) What is the ratio of the skin depth in Copper at 1 KHz to that at 100 MHz?

(b) What is the electric field associated with a laser beam having an energy density 10^6 joules/cm^3?

(c) A radio station 10 miles away radiates 50 watts of vertically polarized radio waves. What is the maximum instantaneous voltage driving the electrons in your receiving antenna if the antenna is 20 cm long and is oriented vertically? Neglect all reflections of the waves from the ground, buildings, etc.

(d) If a candle had the same intrinsic brightness as the moon, it would provide the same illumination as the moon when placed at a distance at which it subtended the same solid angle at your eye as the moon does. Assume that full moonlight provides "perfect illumination for romance". Further, assume that the candle flame has a spherical surface and assume a reasonable value for the "effective" diameter (say 1cm). Calculate how far away the candle should be to provide "perfect illumination" as defined above. The surface brightness of the moon is 0.25 candle/cm^2 while that for the candle is 0.5 candle/cm^2. (1 cd $=$ 1 candle $=$ 20 milliwatts of visible light). The angular diameter of the moon is 1/100 radian.

(e) Suppose right-handed circularly polarized light (by the angular momentum convention) is incident on an absorbing slab. The slab is suspended by a vertical thread. The light is directed upward and hits the underside of the slab. If the circularly polarized light beam has a power of 1 watt at the visible light wavelength of 5500 Å and if all of this light is absorbed by the slab, what is the torque exerted on the slab? (Give the answer in dyne-cm).

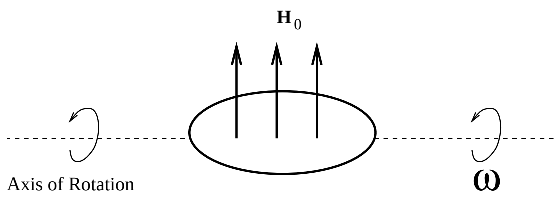
[12]

3. (a) For electric and magnetic fields, show that $\mathbf{B}^2 - \mathbf{E}^2$ and $\mathbf{B}\cdot\mathbf{E}$ are invariant under changes of co-ordinates and Lorentz transformations. Are there any invariants which are not merely algebraic combinations of these two? Prove your answer.

(b) A particular electromagnetic field has its **E** field at an angle θ_0 to its **B** field, and θ_0 is invariant to all observers. What is the value of θ_0? Prove your answer.

(c) Show that $\mathcal{E}^2 - |\mathbf{S}|^2$ is a Lorentz invariant of the electromagnetic field, where $\mathcal{E}$ is the energy density and **S** is the Poynting flux. [8]

4. A thin copper ring rotates about an axis perpendicular to a uniform magnetic field $\mathbf{H}_0 = 200$ gauss. Its initial frequency of rotation is ω_0. Calculate the time it takes the frequency to decrease to $1/e$ of its original value under the assumption that the energy goes into Joule heat. (Copper has conductivity $\sigma = 5 \times 10^{17}$ cgs units and density $\rho = 8.9\,\mathrm{gm/cm^3}$).



[Hint: Consider the average power loss over one cycle to simplify matters]

[12]

5. Would it be necessary to change the focus of the telescope to look at the moon after focusing it on a star? How big would the telesope have to be for any refocussing to be needed? The moon's distance from the earth is 3.8×10^8 meters. Consider the effects of atmospheric turbulence [which limits the resolution to about one arc second] and the limitation due to diffraction effects. Assume the wavelength of light to be 5×10^{-7} meters.

[12]

6. All of you know why the sky appears blue. Here is an argument that seems to show that the sky should be invisible at least in certain directions. Consider a given monochromatic component of sunlight. The electric field drives a given air molecule. Each oscillating charge of the air molecule radiates waves in all directions, some of which travel to the eye of a given observer. But, for a given molecule (call it No.1) there is another (No.2) that is one half-wavelength farther from the observer. If both molecules are driven with the same amplitude and phase constant, their waves should superpose to give zero at the position of the observer. For scattering near 90^0, we can obviously satisfy these phase and amplitude conditions, provided the number of air molecules per unit volume is large enough so that there is nearly always a molecule "No.2" for every molecule "No.1". (For scattering near zero degrees, molecules that are a half-wavelength farther apart from the observer are excited half a period earlier; therefore destructive interference may not be seen). For air at STP, the number density is about $3 \times 10^{19}\,\mathrm{cm}^{-3}$. Therefore a cube of edge length $5 \times 10^{-5}\,\mathrm{cm}$ (which is wavelength of blue light) contains about 4×10^6 molecules, *i.e.* about 100 molecules along each edge of a cube with edge length equal to one wavelength. That would seem to be more than sufficient to give almost completely destructive interference, even when we take into account the fact that the air density falls off exponentially with height above the surface of the earth. Therefore we arrive at the prediction that the part of the sky which corresponds to scattering through about 90 degrees should be "black" rather than bright blue.

This prediction obviously contradicts experience. In fact, the intensity observed is very nearly that which would be predicted on the basis of the scattering from individual air molecules, taking into account the number density of molecules and adding the intensities contributed by independent molecules.

If instead of air one uses glass or clean water, the predicted destructive interference for 90-degree scattering *does* occur. In the case of water, one adds the *amplitudes* for 90-degree scattering, and so one gets the expected destructive interference while in the case of air, it is the intensities which seem to add. Why does the predicted destructive interference not occur for air but occur for water?

[Hint: Consider a small region in space (region 1) and consider another region (region 2) of the same size, situated at the same distance from the sun and situated one half-wavelength (consider a single monochromatic component of sunlight) farther from the observer than region 1. Assume that these regions are small compared with the wavelength. Let there be n_1 and n_2 molecules in regions 1 and 2 respectively. Compute the total electric field due to these regions and average the amplitude over a single period. Now consider the effect of fluctuations in the number of molecules over a long enough period of time. Take it from there.]

[12]

7. (a) Light is normally incident on the interface between two media labelled **1** and **2** respectively. A wave of unit amplitude in medium **1**, incident on the interface, is found to have reflected and transmitted amplitudes r and t respectively. Similarly when the wave is incident in medium **2**, the reflected and transmitted amplitudes are found to be r' and t'. Using the superposition principle and time-reversal invariance, derive the Stokes' relations

$$r^2 + tt' = 1 \qquad \text{and} \qquad r = -r'.$$

(b) A plane electromagnetic wave is normally incident from vacuum on a plane surface film of uniform thickness d covering a semi-infinite dielectric. Assume that $\mu = 1$ for both media and that the film and substrate have indices of refraction n_1 and n_2 respectively. Find an expression for the wave reflected into the vacuum in terms of n_1, n_2 and vacuum wavelength λ. Under what conditions will the reflected wave vanish?

[22]

8. Consider a dielectric nonpolar crystalline material with a simple cubic lattice structure. The lattice constant is given to be 2.00 Å and the refractive index (say, for sodium light) is $n = 2.07$. Suppose that the medium is subjected to a stress leading to a 2% elongation along one cube edge and a 1% contraction along the other two cube edges.

Calculate the refractive index of the strained medium when the electric field vector $\mathbf{E}$ is (a) parallel to and (b) normal to the principal strain axis (along which the strain is maximum). Consider the atomic polarizability α to be a constant scalar quantity.

[Hint: The local field acting on an atom in the medium described above can be found in the following way. Imagine a spherical cavity about the atom, enclosing the six nearest neighbouring atoms. Outside the cavity, the medium may be regarded as continuous and isotropic. The local field $\mathbf{E}_L$ at the center of the cavity may be given by the following expression

$$\mathbf{E}_L = \mathbf{E} + \mathbf{E}' + \sum_{j=1}^{6} \mathbf{E}''_j$$

where $\mathbf{E}$ is the applied field, $\mathbf{E}'$ is the contribution from the polarized component outside the spherical cavity, and $\mathbf{E}''_j$ is the contribution from the dipole induced in the jth atom within the cavity. In an anisotropic medium, $\sum_{j=1}^{6} \mathbf{E}''_j$ does not vanish and it also depends on the direction of the applied $\mathbf{E}$. The field of the atoms outside the cavity $\mathbf{E}'$ is calculated from the average macroscopic polarization density $\mathbf{P}$ by assigning surface charge densities $\sigma = \mathbf{P} \cdot \hat{\mathbf{n}}$ and a volume charge density $\rho = -\nabla \cdot \mathbf{P}$ to the medium. Thus $\mathbf{E}'$ is made up of a contribution from the cavity wall, other surface boundaries in the dielectric (call this contribution $\mathbf{E}_s$) and a contribution from the volume density (call this $\mathbf{E}_v$). Now assume that $\mathbf{P}$ is constant in the immediate neighborhood of the cavity. Then you can identify $\mathbf{E} + \mathbf{E}_s + \mathbf{E}_v$ with the average macroscopic field $\mathbf{E}_{mac}$. Doing this, you should finally get

$$\mathbf{E} + \mathbf{E}' = \mathbf{E}_{mac} + \frac{4\pi}{3}\mathbf{P}.$$

In cases (a) and (b) it is simple to calculate the field due to the induced dipole moment at the center. You should be able to do the rest]

[30]