Midterm Examination

<u>To be returned before 12 noon on October 24</u>

General Instructions

(1) The questions below have a total weightage of 130 marks and your score will be treated as one obtained out of 100. For instance, if you get a score of 85 out of 130, we will actually treat it as 85/100 . All scores above 100 will be set to 100.

(2) You are free to discuss the questions among yourselves if you feel and consult any books in the library. But as in the assignments, your answers should be written in your own words and you should be able to explain every step if asked to do so.

(3) You are free —and even encouraged— to look at standard books in the library for finding answers to the questions. Once again, make sure you understand the answer you are writing.

(4) We would prefer your not discussing the questions with senior colleagues but if you decide to do so, tell the person that this question is a part of a take–home–midterm exam. Making someone give you the answer without giving the person this info will be considered a violation of the “honor-code”.

(5) Most importantly, don’t panic if you find the test a bit difficult! Remember, you need not pass this midterm exam and it only contributes 20% weightage to the final score. We have kept it as a challenge for you to see how well you perform. Also, we have given questions worth 130 marks and increased the time limit to 24 hours.

(6) Needless to say, contact srini if you have doubts about any questions and state clearly in your paper if you think any question is wrong/incomplete.

(7) The time limit will be strictly adhered to. You get zero if you miss the deadline. Check with srini/me as to where we will be around the time you are returning the paper so that you don’t have to run around looking for us!

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1. (a) Show that the Poynting flux $(\mathbf{E} \times \mathbf{B})/4\pi$ is less in magnitude than the energy density $(\mathbf{E}^2 + \mathbf{B}^2)/8\pi$, save for the exceptional case where both Lorentz invariants $[\mathbf{E} \cdot \mathbf{B}$ and $(\mathbf{E}^2 - \mathbf{B}^2)]$ of the field vanish.

(b) The electromagnetic field is said to be "null" at a space-time point x^k if $F^{ij}(x^k)F_{ij}(x^k)$ and ${}^*F_{ij}(x^k)F^{ij}(x^k)$ both vanish at that point. Show that the superposition of two monochromatic plane electromagnetic waves traveling in different directions is null on at most a set of points of measure zero.

(c) In the "generic case" where the field $(\mathbf{E}, \mathbf{B})$ at the point of interest is not null, show that the Poynting flux is reduced to zero by viewing the field from a local inertial frame that is travelling in the direction of $(\mathbf{E} \times \mathbf{B})$ with a velocity

$$v = \tanh \alpha,$$

where the velocity parameter α is given by the formula

$$\tanh 2\alpha = \frac{(\text{Poynting flux})}{(\text{energy density})} = \frac{2\,|\,\mathbf{E} \times \mathbf{B}\,|}{\mathbf{E}^2 + \mathbf{B}^2}.$$

(d) Show that the 4×4 determinant built out of the components of $F_i{}^k$ has the value $-(\mathbf{E} \cdot \mathbf{B})^2$ by evaluating it in the special local inertial frame of part (c).

(e) Regardless of whether the electromagnetic field is or is not null, show that the Maxwell stress–energy tensor has zero trace, *i.e.* $T^i{}_i = 0$, and that its square is multiple of the unit tensor,

$$\begin{aligned} T^i{}_k\, T^k{}_j &= \frac{\delta^i{}_j}{(8\pi)^2}\,[(\mathbf{E}^2 - \mathbf{B}^2)^2 + 4(\mathbf{E} \cdot \mathbf{B})^2] \\ &= \frac{\delta^i{}_j}{(8\pi)^2}\,[(\mathbf{E}^2 + \mathbf{B}^2)^2 - 4(\mathbf{E} \times \mathbf{B})^2]. \end{aligned}$$

[26]

2. The equation of motion (in 4-d notation) of a charged particle in an electromagnetic field can be written in the form

$$\frac{du^i}{ds} = \frac{e}{mc^2}\, F^i{}_k\, u^k$$

For constant electric and magnetic fields the above equation is a matrix equation

$$\frac{du}{ds} = F\,u$$

with u denoting the four vector and F is the constant matrix denoting $F^i{}_k$. The constant factor e/mc^2 has been conveniently set to unity by a suitable redefinition of $F^i{}_k$. This equation has the formal solution

$$u = e^{sF} u_0,$$

where the exponent of a matrix is defined in terms of the power series of the exponential *i.e.*

$$e^{sF} = I + sF + \frac{s^2}{2!} FF + \ldots$$

and $u(s)$ is a column vector made of the components of the four velocity evaluated at the proper time s and $u_0 \equiv u(s = 0)$. Expressions like $\exp(sF)$ can be evaulated easily if one can evaluate the eigenvalues of F. What are the eigenvalues of F?.

One possible way to find the eigenvalues is the following:

(i) Define the quantities

$$\mathcal{F} = \frac{1}{4} F^{ij} F_{ij} = \frac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2) \qquad ; \qquad \mathcal{G} = \frac{1}{4} {}^*F_{ij} F^{ij} = \mathbf{E}\cdot\mathbf{B}.$$

(ii) Next, prove the identities

$$F_{ik} {}^*F^k{}_j = -\delta_{ij}\,\mathcal{G} \qquad ; \qquad {}^*F_{ik} {}^*F^k{}_j - F_{ik} F^k{}_j = 2\delta_{ij}\,\mathcal{F}.$$

(iii) Given the eigenvalue equation $F^i{}_j u^j = f u^i$, show that ${}^*F^i{}_j u^j = -(\mathcal{G}/f)\,u^i$.

(iv) Now find an eigenvalue equation for f in terms of $\mathcal{F}$ and $\mathcal{G}$ and thus compute the eigenvalues.

You can, of course, find f by any other method and full credit will be given. [26]

3. Solve the equations of motion for a charged particle in a plane monochromatic electromagnetic wave travelling along the positive x-axis.

This is most easily done by writing down the Hamilton-Jacobi equation for the action, imposing the Lorentz condition on the vector potential and seeking a solution of the form

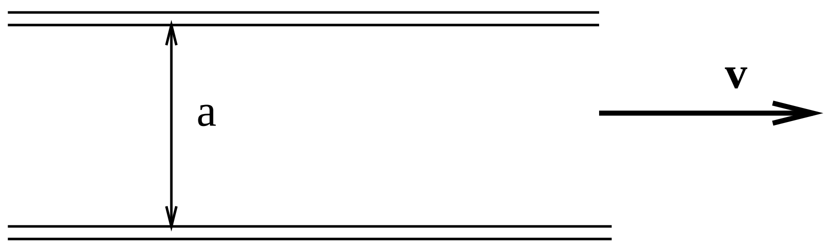
$$S = -f^i x_i + F(ct - x)$$

where S is the action and f^i is a constant four-vector satisfying the condition $f^i f_i = m^2c^2$. What can you say about the energy and momentum transferred to the charged particle? You are, of course, free to solve the equations of motion using any other method. [26]

4. Consider a monochromatic plane electromagnetic wave with a wave vector $k^i = (\omega, \mathbf{k})$ and electric field amplitude $\mathbf{E}_0$. Show that the quantity $|\mathbf{E}_0|/\omega$ is Lorentz invariant. [22]

5. A light beam of intensity I_0 and frequency ν_0 directed along the positive z-axis is reflected normally by a perfect mirror moving along the positive z-axis with a velocity v_0. What is the intensity I and frequency ν of the reflected light in terms of I_0 and ν_0? [10]

6. Two thin, parallel, infinitely long, nonconducting rods, kept a distance a apart , with identical constant charge density λ per unit length in their rest frame, move with a velocity v, not necessarily small compared to the speed of light.



Calculate the force per unit length between them (i) in a frame of reference in which the rods are at rest and (ii) in a frame of reference in which the rods are moving and compare the results. [10]

7. Consider the flow of electrons in a vacuum diode. Assume plane symmetry and ignore edge effects. Let the flow occur only along the x direction. Assume that the electrons leave the cathode with zero initial velocity, that the cathode potential is zero and the anode potential V_0.

Now the electric field is affected by the presence of the electrons and so the potential drop is a function of x. Using Poisson's equation and other physical conditions you feel are relevant obtain a differential equation relating $V(x)$ and x. If one applies the boundary condition for V one obtains a family of solutions. So, impose the following restriction in order to get a unique solution. Consider the limiting case when, for a given V_0, the repulsive effect of the space charge is so great that no more electrons could get across even if more were to be released from the cathode. This is equivalent to the condition that the electric field is zero at the cathode. Show that the relation between the current I and potential V_0 is of the form $I = K\,V_0^{3/2}$ where K is a constant. This is an example of a system that does not obey Ohm's law. [10]